Shape programming of a magnetic elastica

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Talk based on past and present collaborations with
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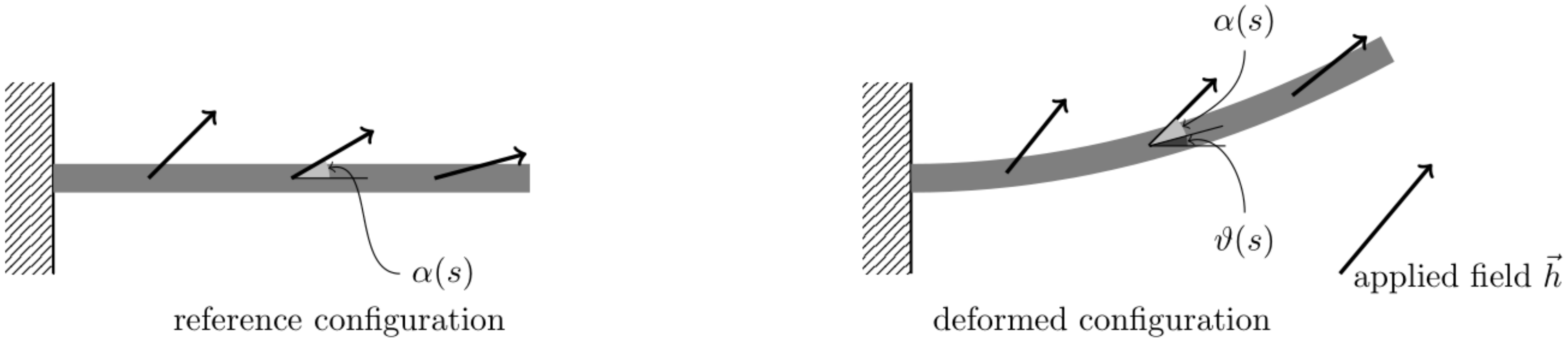
- Established in 1992
- 12 departments
- Enrollment: 35000 students
- Second-largest university of Rome by enrollment and one of the largest research-based institutions in the country.

A magnetoelastic cantilever beam



Given

- applied field $\vec{h} \in \mathbb{R}^2$
- orientation $\alpha : (0,1) \to \mathbb{R}$ of the magnetization

minimize the *magnetoelastic energy*

$$\mathcal{E}(\vartheta) = \int_0^1 \left[\frac{\vartheta'^2}{2} - \vec{h} \cdot \vec{m}(\alpha + \vartheta)\right] ds, \qquad \vec{m}(\varphi) = (\cos\varphi, \sin\varphi).$$

in the admissible space $H^1_{0L} := \{u \in H^1 : u(0) = 0\}$.

Euler-Lagrange system

$$\begin{cases} -\vartheta'' - \vec{h} \cdot D\vec{m}(\alpha + \vartheta) = 0, \\ \vartheta(0) = 0, \\ \vartheta'(1) = 0. \end{cases} \tag{EL}$$

Let

$$c_p := \frac{2}{\pi}$$

be the best constant in the Poincaré inequality:

$$\int_0^1 w^2 \leq c_p \int_0^1 (w')^2, \qquad w \in C^1([0,1]), \quad w(0) = 0.$$

Proposition 1. *The magnetoelastic energy has at least a minimizer. Moreover, this minimizer is a strong solution of* (EL), *and it is unique if*

$$|\vec{h}| \leq c_p^{-2} \tag{B}$$

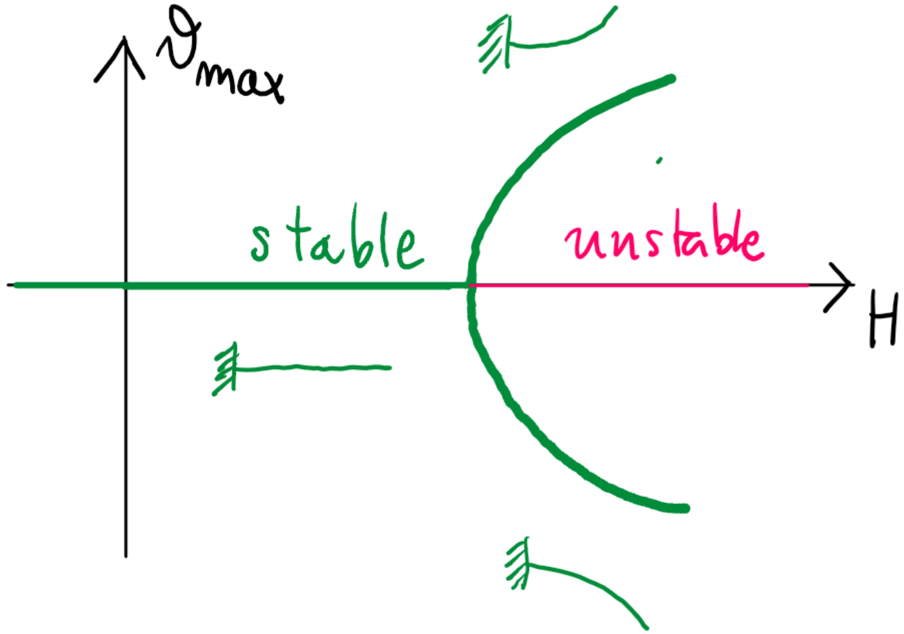
Magnetoelastic instabilities

Take

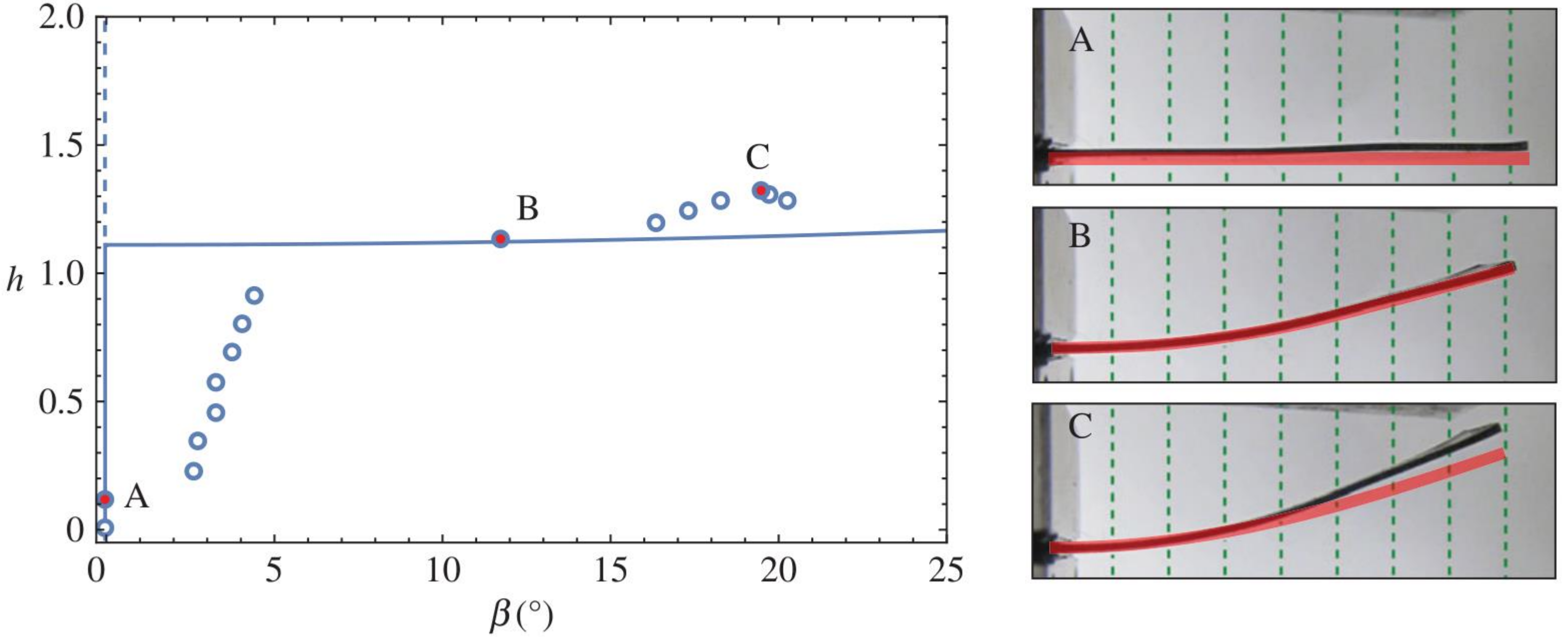
$$\vec{h} = (-h, 0), \qquad \alpha = 0.$$

Then the Euler-Langrange equation becomes

$$-\vartheta'' - H \sin \vartheta = 0.$$



Magnetoelastic instabilities



Stanier DC, Ciambella J, Rahatekar SS. 2016 Fabrication and characterisation of short fibre reinforced elastomer composites for bending and twisting magnetic actuation. *Compos. A: Appl. Sci. Manuf.* **91**, 168–176. (doi:10.1016/j.compositesa.2016.10.001)

RESEARCH ARTICLE

Shape-programmable magnetic soft matter

Guo Zhan Lum, Zhou Ye, Xiaoguang Dong, Hamid Marvi, Onder Erin, Wenqi Hu, and Metin Sitti

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https://doi.org/10.1073/pnas.1608193113

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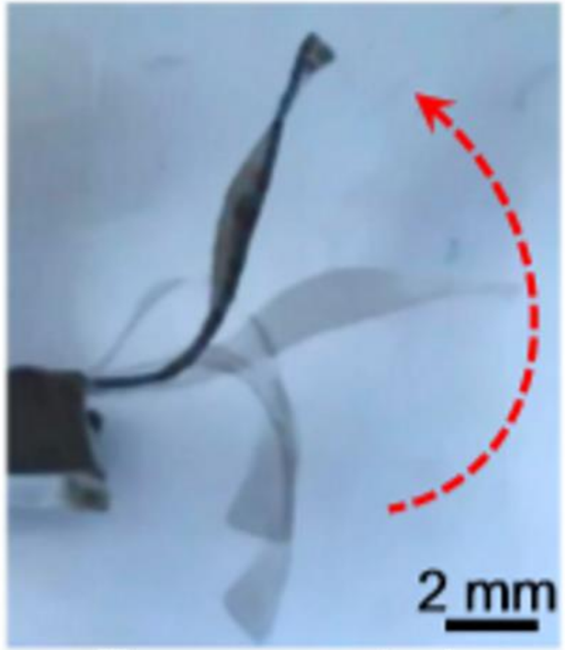
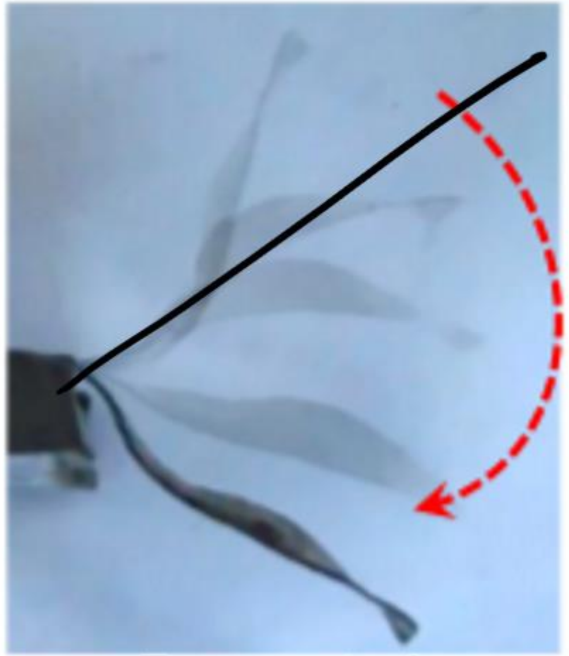
- Jellyfish-like robot
- Undulating swimmer



Recovery stroke


Power stroke

•Artificial cilium

The cost functional

Given a list of *target shapes* $\overline{\boldsymbol{\vartheta}} = (\bar{\vartheta}_1, \ldots, \bar{\vartheta}_n)$ with $\bar{\vartheta}_i \in L^2((0,1))$, define the *regularized cost*:

$$\mathcal{C}(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}) = \frac{1}{2}\sum_{i=1}^{n}\int_0^1 |\vartheta_i - \bar{\vartheta}_i|^2 + \frac{\varepsilon}{2}\int_0^1 |\alpha'|^2 + \frac{\gamma}{2}\sum_{i=1}^{n}|\vec{h}_i|^2 \tag{Cost}$$

in the admissible space

$$\mathcal{H} = \left\{(\overrightarrow{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}) : \overrightarrow{\boldsymbol{h}} \in \mathbb{R}^{2n}, \alpha \in H^1_{0L}, \boldsymbol{\vartheta} \in (H^1_{0L})^n\right\} = \mathbb{R}^{2n} \times H^1_{0L}(I) \times (H^1_{0L})^n.$$

Optimal design-control Problem

Minimize $\mathcal{C}$ in the *admissible set* $\mathcal{A} \subset \mathcal{H}$ of triplets satisfying the *control-to-state* system

$$\begin{cases} -\vartheta_i'' - \vec{h} \cdot D\vec{m}(\alpha + \vartheta_i) = 0 \quad \text{in } (0,1), \\ \vartheta_i(0) = 0, & i = 1, \ldots, n, \\ \vartheta_i'(1) = 0. \end{cases} \tag{P_{ϑ_i}}$$

Existence of a minimizer

Proposition 2 (Existence)**.** *The cost functional has a minimizer* $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta})$ *in the admissible set.*

Remark 2. *Using* $(0,0,0)$ *as comparison we obtain:*

$$\max_i |\vec{h}_i|^2 \leq \frac{\overline{\Theta}^2}{\gamma}, \quad \textit{where} \quad \overline{\Theta}^2 = \sum_i \int \overline{\theta}_i^2. \tag{EOMF}$$

Thus, if the target rotations are small enough, the condition (B) *is fulfilled, and* **mechanical equilibria identified by minimization are stable***.*

What about uniqueness?

Our answer:

- We replace the variational problem with the corresponding Euler system, incorporating the state equations through a list of *Lagrange multipliers*.
- We derive sufficient conditions for the solution of this system to be unique, using a fixed point argument.
- Our proof provides a hint towards a numerical solution strategy.

Tools:

- Characterization of Fréchet-differentiable functions;
- A Lagrange Multiplier Theorem in the Banach-space setting;
- Contraction theorem.

Lagrange-multiplier formulation

Lagrangian

$$\mathcal{L}(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}, \boldsymbol{\lambda}) := \mathcal{C}(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}) - \sum_{i=1}^{n} \int_0^1 \lambda_i(s) \left(-\vartheta_i''(s) - \vec{h}_i \cdot D\vec{m}(\alpha(s) + \vartheta_i(s)) \right) ds$$

where $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$, the *adjoint state*, is a Lagrange multiplier

Existence of a Lagrange multiplier

Proposition 3. *Let* $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta})$ *be an extremal of* $\mathcal{C}$ *in* $\mathcal{A}$ *such that* $\max_i |\vec{h}_i| < c_p^{-2}$. *Then there exists a Lagrange multiplier* $\boldsymbol{\lambda} \in H^1_{0L}(I)^n$ *such that* $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}, \boldsymbol{\lambda})$ *is a* **strong** *solution of the* E-L system:

$$\begin{cases} (P_{\vartheta_i}): \quad -\vartheta_i'' - \vec{h}_i \cdot D\vec{m}(\alpha + \vartheta_i) = 0, & \vartheta_i(0) = \vartheta_i'(1) = 0 \\ (P_{\lambda_i}): \quad -\lambda_i'' - \lambda_i \vec{h}_i \cdot D^2\vec{m}(\alpha + \vartheta_i) = \vartheta_i - \bar{\vartheta}_i, & \lambda_i(0) = \lambda_i'(1) = 0 \\ (P_{\alpha}): \quad -\varepsilon\alpha'' + \sum_{i=1}^{n} \lambda_i \vec{h}_i \cdot D^2\vec{m}(\alpha + \vartheta_i) = 0, & \alpha(0) = \alpha'(1) = 0 \\ (P_{\vec{h}}): \quad \vec{h}_i = -\frac{1}{\gamma}\int_0^1 \lambda_i D\vec{m}(\alpha + \vartheta_i) & \end{cases} \tag{3}$$

Uniqueness

Theorem. *For every list* $\overline{\boldsymbol{\vartheta}}$ *of target shapes, for every* $\varepsilon > 0$*, and for every* $K < c_p^{-2}$ *there exists* $\gamma^* = \gamma^*(\overline{\boldsymbol{\vartheta}}, \varepsilon, K)$ *such that for every* $\gamma > \gamma^*$ *there exists a unique solution* $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}, \boldsymbol{\lambda}) \in \mathcal{H} \times H^1_{0L}(I)^n$ *of the EL-system such that*

$$\max_i |\vec{h}_i| \leq K.$$

Corollary. *Let* $\overline{\boldsymbol{\vartheta}} \in C([0,1])^n$*,* $\varepsilon > 0$*, and let* $0 < K < c_P^{-2}$*. Then there exists* $\gamma_{**} = \gamma_{**}(\overline{\boldsymbol{\vartheta}}, \varepsilon, K)$ *such that for any* $\gamma > \gamma_{**}$ *the minimizer is unique. Furthermore, it coincides with the unique solution of the EL-system, whence it is smooth and such that* $\alpha'(0) = 0$*.*

The uniqueness argument

1) Let $D = \{\vec{\boldsymbol{h}} : \max_i |\vec{h}_i| \leq K\}$. We know that if $K < c_p^{-2}$ the following map is well defined:

$$\vec{\boldsymbol{h}}^{(k)} \overset{(P_{\boldsymbol{\vartheta},\boldsymbol{\lambda}})}{\mapsto} (\boldsymbol{\vartheta}^{(k)}, \boldsymbol{\lambda}^{(k)}) \overset{(P_{\vec{\boldsymbol{h}}})}{\mapsto} \vec{\boldsymbol{h}}^{(k+1)}.$$

Using the estimates (7)–(8) we show that this map is a **contraction** in D, and hence it has a **unique** fixed point. Thus, the solution operator

$$\alpha \overset{(P_{\boldsymbol{\vartheta},\boldsymbol{\lambda},\vec{\boldsymbol{h}}})}{\mapsto} (\vec{\boldsymbol{h}}(\alpha), \boldsymbol{\vartheta}(\alpha), \boldsymbol{\lambda}(\alpha))$$

is well defined.

The uniqueness argument (cont.d)

2) We show that the solution operator $\alpha_k \overset{(\widetilde{P})_\alpha}{\mapsto} \alpha_{k+1}$ of the problem

$$(\widetilde{P}_\alpha) \qquad \begin{cases} -\varepsilon\alpha''_{k+1} + \sum_{i=1}^{n} \lambda_i(\alpha_k)\vec{h}_i(\alpha_k) \cdot D^2\vec{m}(\alpha_k + \vartheta_i(\alpha_k)) = 0, \\ \alpha_{k+1}(0) = 0, \\ \alpha'_{k+1}(1) = 0. \end{cases}$$

is a contraction.

A numerical scheme

Initialisation:

$\alpha \longleftarrow$ initial guess $\alpha^{(0)}$;

$\vec{h}_i \longleftarrow$ initial guess $\vec{h}_i^{(0)}$, $i = 1, \ldots n$;

$\lambda_i \longleftarrow$ initial guess $\lambda_i^{(0)}$, $i = 1, \ldots n$;

$tol \longleftarrow$ tolerance;

repeat

repeat

$\vartheta_i \longleftarrow$ solve (P_{ϑ_i}), $i = 1, \ldots, n$;

$\lambda_i \longleftarrow$ solve (P_{λ_i}), $i = 1, \ldots, n$;

$\vec{h}_i^{old} \longleftarrow \vec{h}_i$;

$\vec{h}_i \longleftarrow$ solve $(P_{\vec{h}_i})$, $i = 1, \ldots, n$;

until $\max_{i=1}^{n} |\vec{h}_i^{old} - \vec{h}_i| \leq tol$;

$\alpha^{old} \longleftarrow \alpha$;

$\alpha \longleftarrow$ solve (P_α);

until $\|\alpha^{old} - \alpha\|_\infty \leq tol$;

Properties of the E-L system.

Remark 5 (Key for the fixed-point argument). *System* $(P_{\vartheta_i}, P_{\lambda_i}, P_{\vec{h}_i})$ *and* $(P_{\vartheta_j}, P_{\lambda_j}, P_{\vec{h}_j})$ *for* $i \neq j$ *are coupled* **only** *through* (P_α).*

Remark 6 (Key for the estimates). *Problems* (P_θ), (P_λ) *and* (P_α) *share the same structure:*

$$\int_0^1 v'w' + \int_0^1 f(s,v)w = 0, \qquad \textit{for all } w \in H^1_{0L}(I). \tag{4}$$

where $f(s,\cdot)$ *has Lipschitz constant*

$$L \leq |\vec{h}_i|.$$

Proposition 6. *If* $L < c_p^{-2}$ *then* (4) *has* **unique** *solutio* v *with the bounds*

$$\|v\| \leq \frac{c_p}{1 - Lc_p^2}\|f_0\|_{L^2}, \quad \|v\|_\infty \leq \|f\|_\infty. \tag{5}$$

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*This reflects the fact that once the design α is given each optimization problem represents a different physical experiment.

Existence of a Lagrange multiplier: tools

Remark 4. *The admissible space $\mathcal{A}$ is the null set of the* constraint mapping $G : \mathcal{H} \to (H^1_{0L}(I)^n)'$ *defined by*

$$\langle G(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}), \boldsymbol{u} \rangle = \sum_{i=1}^{n} \left\{ \int_0^1 \vartheta_i' u_i' - \int_0^1 \vec{h} \cdot D\vec{m}(\alpha + \vartheta_i) u_i \right\} \qquad \textit{for all } \boldsymbol{u} \in H^1_{0L}(I)^n.$$

Proposition 4. *The cost functional $\mathcal{C} : \mathcal{H} \to \mathbb{R}$ and the constraint mapping $G : \mathcal{H} \to (H^1_{0L}(I)^n)'$ are Fréchet differentiable everywhere.*

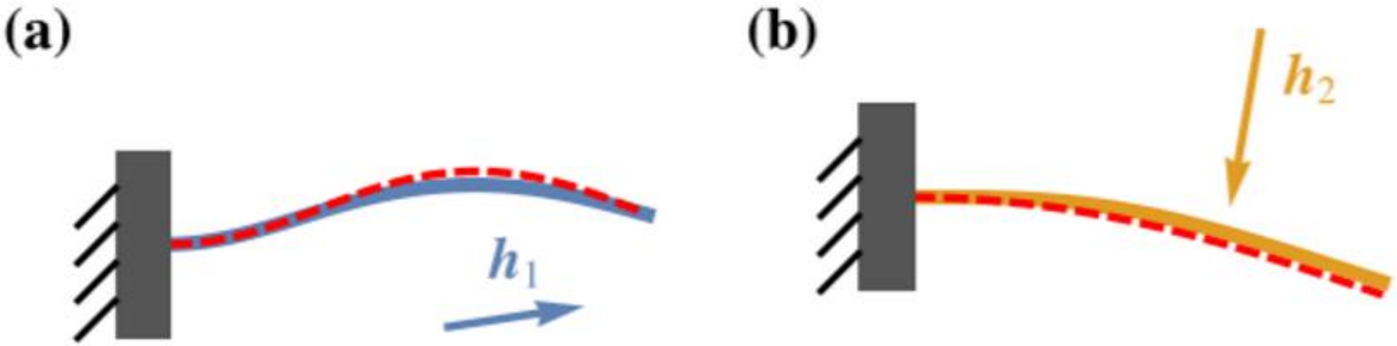
Proposition 5. *If $\max_i |\vec{h}_i| < c_p^{-2}$ then $DG(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}) : \mathcal{H} \to (H^1_{0L}(I)^n)'$ is surjective, i.e., $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta})$ is a* regular point *of G.*

We obtain Proposition 3 by combining the above facts with:

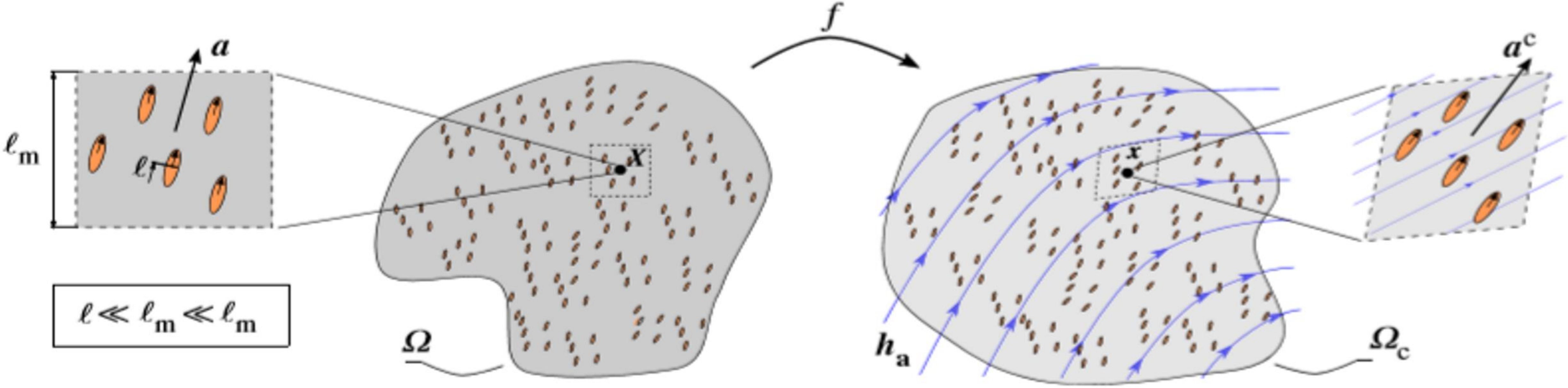
Theorem 1 (Lagrange multiplier theorem)**.** *Let $f : \mathcal{U} \subset \mathcal{X} \to \mathbb{R}$ and $G : \mathcal{U} \subset \mathcal{X} \to \mathcal{Y}$ be C^1 on an open neighborhood $\mathcal{U}$ of $\tilde{x}$. Suppose that $\tilde{x}$ is an extremum of f on the set $\{x \in \mathcal{U} : G(x) = 0\}$ and that $\tilde{x}$ is a regular point of G. Then there exists a Lagrange multiplier $\lambda \in \mathcal{Y}'$ such that*

$$Df(\tilde{x}) - \langle \lambda, DG(\tilde{x}) \rangle = 0.$$

Remark. *Use $W^{2,2}$ regularity to obtain strong solutions.*


(a)
h1
(b)
h2


h1
h2



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