Weakly nonlinear analysis

The weakly nonlinear analysis is a procedure to approximate a branch of a bifurcation diagram.

To fix ideas, suppose that we have a potential energy $W(u, \lambda)$, where λ is the loading parameter, and u is the unknown. We assume that $u = 0$ is the fundamental branch, so that $D_u W(0, \lambda) = 0$ for every λ.

Any bifurcated branch can be represented as a parametrized curve $\xi \mapsto (u_b(\xi), \lambda_b(\xi))$. This branch satisfies the virtual work equation:

$$D_u W(u_b(\xi), \lambda_b(\xi))[\tilde{u}] = 0, \qquad \text{for every virtual variation } \tilde{u}, \tag{1}$$

(here we think of the Frechet derivative of W as a linear form acting on virtual variations) and is assumed to cross the fundamental branch $u = 0$ at $(0, \lambda_c)$. Weakly nonlinear analysis assumes that the solution branch admits the expansion:

$$\begin{aligned} u_b(\xi) &= \xi u_1 + \xi^2 u_2 + \dots, \\ \sigma_b(\xi) &= \lambda_c + \xi \lambda_1 + \xi^2 \lambda_2 + \dots, \end{aligned}$$

and extracts information about the coefficients by plugging this expansion into (1).

As a consequence of the above ansatz, the function

$$F(\xi) = D_u W(u_b(\xi), \lambda_b(\xi)) \tag{2}$$

can be written in power-series expansion:

$$F(\xi) = F_0 + \xi F_1 + \xi^2 F_2 + \dots, \tag{3}$$

where the coefficients F_i depend on the derivatives of W at $(0, \lambda_c)$ and on the coefficients in the expansion of u_b and λ_b. The equations

$$\lambda_1 = -\frac{1}{2} \frac{D_u^3 W(0, \lambda_c)[u_1, u_1, u_1]}{D_\lambda D_u^2 W(0, \lambda_c)[u_1, u_1]}, \tag{4}$$

and

$$\lambda_2 = -\frac{1/6 D_u^4 W(0, \lambda_c)[u_1, u_1, u_1, u_1] + D_u^3 W(0, \lambda_c)[u_1, u_1, u_1]}{D_\lambda D_u^2 W(0, \lambda_c)[u_1, u_1]}. \tag{5}$$

From the practical point of view, the calculation of the derivatives can be carried out easily through the formula:

$$D_u^n W(0, \lambda_c)[u_1, \dots, u_1] = \left. \frac{\mathrm{d}^n}{\mathrm{d}\eta^n} \right|_{\eta=0} W(\eta u_1, \lambda_c) \tag{6}$$