

A JUSTIFICATION OF THE REISSNER–MINDLIN PLATE THEORY THROUGH VARIATIONAL CONVERGENCE

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We provide a justification of the Reissner–Mindlin plate theory, using linear three-dimensional elasticity as framework and Γ -convergence as technical tool. Essential to our developments is the selection of a transversely isotropic material class whose stored energy depends on (first and) second gradients of the displacement field. Our choices of a candidate Γ -limit and a scaling law of the basic energy functional in terms of a thinness parameter are guided by mechanical and formal arguments that our variational convergence theorem is meant to validate mathematically.

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1. Introduction

This paper is a reasonably detailed exposition of results recently announced in [7]. The first part, which consists of Secs. 2, 3, and 4, is meant to set the stage for the developments of Sec. 5, where we present a validation of the two-dimensional Reissner–Mindlin plate theory as the variational limit of an appropriate three-dimensional *parent theory* of linear elasticity.

In Sec. 2, which is partly based on results detailed in [8, 9], we argue that such parent theory should have two nonclassic features: firstly, the material response should be transversely isotropic; secondly, the stored-energy density should depend on both the first and the second gradient of displacement. The mechanical and formal arguments we offer are meant to support our mathematical choice of the main

ingredients of a Γ -convergence proof: a scaled family of parent energy functionals (as suggested by the scaling procedure proposed in [5,6] and recapitulated in Sec. 3) and a candidate Γ -limit (as suggested by the heuristic reasoning developed in Sec. 4), satisfying a “liminf inequality” and a “recovery-sequence condition”, the latter for sequences converging to Reissner–Mindlin displacement fields.

In Sec. 5, we prove within a precise functional analytic framework that the kinematic *Ansatz* implicit in the Reissner–Mindlin theory can be recovered by a compactness argument (Lemma 5.1); and that (Theorem 5.3) the candidate Γ -limit can be validated by using the recovery sequence constructed in Sec. 4. Our main findings are wrapped up in the final Sec. 6.

2. A Survey of the Information Extractible from the Reissner–Mindlin Theory

In this section, we discuss those features of the Reissner–Mindlin plate theory that we regard as useful background information for whatever three-dimensional justification one may wish to furnish for it.

Reissner–Mindlin’s theory is a linear one for the equilibrium of elastic cylinders of moderate thickness, a theory that can be formulated as a minimum problem ruled by an energy functional having the form of an integral over a two-dimensional flat region, identified with the mid cross-section of the cylindrical plate-like body under study. We hereafter introduce the quadratic principal part of this functional and illustrate the three-dimensional interpretation of the unknown fields in terms of which it is expressed. Such an interpretation, as we shall see, is expedient to put us on the right track and establish the relationship of the Reissner–Mindlin functional and its minima with an appropriate family of three-dimensional energy functionals, their Γ -limit, and the minima of the latter.

2.1. The Reissner–Mindlin flexure functional

Let $\{o; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a cartesian reference of origin o and orthonormal base vectors \mathbf{e}_i , let (x_1, x_2, x_3) be the associated coordinates of a typical space point, and let \mathcal{P} be a simply-connected regular region of the plane $x_3 = 0$.

A *Reissner–Mindlin plate* occupying \mathcal{P} is a two-dimensional material body having *effective thickness* t , *shearing stiffness* $S = S_0 t$, *bending stiffness* $B = B_0 t^3$, and *lateral contraction modulus* ν ; the parameter t is geometric, the constitutive parameters S_0, B_0 , and ν are chosen so as to guarantee positivity of the *flexure-energy functional*

$$\Sigma_{RM}(w, \boldsymbol{\varphi}) = \frac{1}{2} \int_{\mathcal{P}} (S|\nabla w + \boldsymbol{\varphi}|^2 + B((1 - \nu)|\mathbf{E}(\boldsymbol{\varphi})|^2 + \nu(\operatorname{div} \boldsymbol{\varphi})^2)), \quad (2.1)$$

where $w = w(x_1, x_2)$ and $\boldsymbol{\varphi} = \sum_{\alpha=1}^2 \varphi_{\alpha}(x_1, x_2)\mathbf{e}_{\alpha}$ are, respectively, a scalar field and a plane vector field on \mathcal{P} , and where

$$\mathbf{E}(\boldsymbol{\varphi}) = \operatorname{sym} \nabla \boldsymbol{\varphi} = \frac{1}{2}(\nabla \boldsymbol{\varphi} + \nabla \boldsymbol{\varphi}^T).$$

The *flexure equilibria* of a Reissner–Mindlin plate are solutions of the Euler–Lagrange equations associated with a *total energy* functional Π_{RM} , whose principal quadratic part is Σ_{RM} in (2.1) and whose linear part is the *load potential* (to be introduced in Sec. 3).

2.2. The first-gradient plate functional

The term “effective thickness”, as well as the rest of the terminology in italics in the preceding subsection, alludes to the following three-dimensional interpretation of the Reissner–Mindlin kinematics, and hence of the solutions of the two-dimensional variational problems governed by the functional Π_{RM} .

For \mathcal{C} , a right cylinder of cross section \mathcal{P} and height $2h$, and for $\mathbf{u} = \sum_{i=1}^3 u_i(x_1, x_2, x_3)\mathbf{e}_i$, a vector field defined on \mathcal{C} , consider the *first-gradient plate functional*

$$\Sigma_{(1)}(\mathbf{u}) = \int_{\mathcal{C}} \sigma_{(1)}(\mathbf{E}(\mathbf{u})), \tag{2.2}$$

where

$$\begin{aligned} \sigma_{(1)}(\mathbf{E}) &= 2\gamma(E_{13}^2 + E_{23}^2) \\ &+ \frac{E}{2(1-\nu^2)}((E_{11} + E_{22})^2 - 2(1-\nu)(E_{11}E_{22} - E_{12}^2)) \end{aligned} \tag{2.3}$$

with the material moduli γ, E , and ν (all to be given a physical interpretation later on) such that the stored-energy density $\sigma_{(1)}$ is positive. If one takes $S_0 = \gamma$, $B_0 = \frac{E}{12(1-\nu^2)}$, and $t = 2h$, then

$$\Sigma_{(1)}(\mathbf{u}_{RM}) = \Sigma_{RM}(w, \boldsymbol{\varphi}) \tag{2.4}$$

for

$$\mathbf{u}_{RM}(x_1, x_2, x_3) = w(x_1, x_2)\mathbf{e}_3 + x_3\boldsymbol{\varphi}(x_1, x_2), \quad \boldsymbol{\varphi} \cdot \mathbf{e}_3 = 0, \tag{2.5}$$

the *Reissner–Mindlin displacement field* in \mathcal{C} . The vector field \mathbf{u}_{RM} is parametrized by the *cross-section displacement* w in the direction \mathbf{e}_3 and the *transverse-fiber rotation* $\boldsymbol{\varphi}$. The transverse fiber through a point $x \equiv (x_1, x_2)$ of \mathcal{P} is the segment $\{p \in \mathcal{C} \mid p = x + x_3\mathbf{e}_3, x_3 \in [-h, +h]\}$; when \mathcal{C} undergoes a Reissner–Mindlin displacement, that typical transverse fiber rotates about x as prescribed by the vector $\boldsymbol{\omega} = \mathbf{e}_3 \times \boldsymbol{\varphi}$, where $\boldsymbol{\omega}$ is such that $x_3\boldsymbol{\varphi} = \boldsymbol{\omega} \times (p - x)$.

Thus, (i) the Reissner–Mindlin kinematics for the flat region \mathcal{P} is in one-to-one correspondence with a special displacement class for the cylinder \mathcal{C} ; (ii) for each solution of a two-dimensional equilibrium problem for a Reissner–Mindlin plate occupying \mathcal{P} , there is an equi-energetic Reissner–Mindlin displacement field in the plate-like elastic body of stored-energy $\sigma_{(1)}$ occupying \mathcal{C} ; (iii) according to a presumption common to all thin-structure theories, as the thickness-to-diameter ratio of \mathcal{C} gets smaller and smaller, such a Reissner–Mindlin field should approximate

better and better in a suitable energy norm the unknown solution of a three-dimensional equilibrium problem posed on \mathcal{C} , whose stored energy was of type (2.3) and whose data would agree in a suitable global sense with those of the two-dimensional problem in question.

2.3. Hints about the choice of a three-dimensional material response

The three-dimensional material response implied by formula (2.3) for the stored-energy density is a special instance of linearly elastic, transversely isotropic response with respect to the direction \mathbf{e}_3 . In fact, in terms of stress \mathbf{S} and strain \mathbf{E} , the unconstrained \mathbf{e}_3 -transversely isotropic material response may be given the general form:

$$\begin{aligned} S_{\alpha\beta} &= 2\mu E_{\alpha\beta} + (\lambda(E_{11} + E_{22}) + \tau_2 E_{33})\delta_{\alpha\beta}, \\ S_{33} &= \tau_1 E_{33} + \tau_2(E_{11} + E_{22}), \quad S_{3\alpha} = 2\tau_3 E_{3\alpha}; \end{aligned} \tag{2.6}$$

and the corresponding stored-energy density is

$$\begin{aligned} \sigma_{t-i}(\mathbf{E}) &= \frac{1}{2}\tau_1(E_{33})^2 + \tau_2(E_{11} + E_{22})E_{33} + 2\tau_3(E_{13}^2 + E_{23}^2) \\ &\quad + \frac{1}{2}(\lambda + 2\mu) \left((E_{11} + E_{22})^2 - \frac{4\mu}{\lambda + 2\mu}(E_{11}E_{22} - E_{12}^2) \right), \end{aligned} \tag{2.7}$$

subject to the positivity conditions:

$$\tau_1 > 0, \quad \tau_3 > 0, \quad \mu > 0, \quad \hat{\lambda} + \mu > 0, \quad \text{with } \hat{\lambda} := \lambda - \tau_2^2/\tau_1. \tag{2.8}$$

Furthermore, the Reissner–Mindlin displacement field (2.5) is the general solution of a system of partial differential equations, namely,

$$u_{3,3} = 0, \quad u_{1,33} = 0, \quad u_{2,33} = 0 \tag{2.9}$$

(to within a plane displacement field $\mathbf{v} = \sum_{\alpha} v_{\alpha}(x_1, x_2)\mathbf{e}_{\alpha}$ that we here ignore to concentrate on flexure deformations). These partial differential equations express two *internal constraints* on the admissible deformations: the first, *inextensibility* in the direction \mathbf{e}_3 , a standard first-order constraint; the second and third, linearity with respect to the coordinate x_3 of the in-plane components of the displacement field, a less familiar second-order constraint.

These observations prompt the conjecture that a three-dimensional parent theory of linear elasticity allowing for validation by variational convergence of the two-dimensional Reissner–Mindlin theory should be transversely isotropic and consistent, perhaps only in the limit of vanishing thickness, with the internal constraints (2.9). Such conjecture is reinforced by the results of the formal scaling procedure proposed in [6] and recapitulated in Sec. 3, and by the heuristic argument in Sec. 4, pointing to the choice of a Γ -limit coded in the form of the first-order plate functional (2.2)–(2.3).

2.4. Two alternative interpretations of the first-gradient plate functional

There are *two different ways* to regard the stored-energy density $\sigma_{(1)}$ in (2.3) as a special instance of the general stored-energy density (2.7). Such a double interpretation is granted by accounting in two different ways for the kinematical constraint of inextensibility encoded in (2.9)₁:

$$u_{3,3} = 0 \Leftrightarrow E_{33}(\mathbf{u}) = \mathbf{E}(\mathbf{u}) \cdot \mathbf{V} = 0, \quad \mathbf{V} := \mathbf{e}_3 \otimes \mathbf{e}_3. \quad (2.10)$$

Consider the following stored energies, each of which is derived from (2.7):

$$\check{\sigma}_{(1)} := \sigma_{t-i}|_{\mathbf{E} \cdot \mathbf{V} = 0} \quad \text{and} \quad \hat{\sigma}_{(1)} := \sigma_{t-i}|_{\mathbf{S} \cdot \mathbf{V} = 0}. \quad (2.11)$$

As a direct computation shows,

$$\begin{aligned} \check{\sigma}_{(1)}(\mathbf{E}) &= 2\tau_3(E_{13}^2 + E_{23}^2) \\ &\quad + \frac{1}{2}(\lambda + 2\mu) \left((E_{11} + E_{22})^2 - \frac{4\mu}{\lambda + 2\mu}(E_{11}E_{22} - E_{12}^2) \right) \end{aligned} \quad (2.12)$$

and

$$\begin{aligned} \hat{\sigma}_{(1)}(\mathbf{E}) &= 2\tau_3(E_{13}^2 + E_{23}^2) \\ &\quad + \frac{1}{2}(\hat{\lambda} + 2\mu) \left((E_{11} + E_{22})^2 - \frac{4\mu}{\hat{\lambda} + 2\mu}(E_{11}E_{22} - E_{12}^2) \right), \end{aligned} \quad (2.13)$$

with $\hat{\lambda}$ given by the last relation in (2.8). Since

$$\sigma_{t-i}|_{\mathbf{S} \cdot \mathbf{V} = 0}(\mathbf{E}) = \min\{\sigma_{t-i}(\mathbf{E} + g\mathbf{V}) : g \in \mathbb{R}\}, \quad (2.14)$$

we have that

$$\check{\sigma}_{(1)}(\mathbf{E}) \geq \hat{\sigma}_{(1)}(\mathbf{E}) \quad (2.15)$$

(with equality, for every \mathbf{E} , when $\tau_2 = 0$). Thus, given a constraint tensor \mathbf{V} compatible with the response symmetry typical of a given material class, restriction to the relative admissible stresses (those such that $\mathbf{S} \cdot \mathbf{V} = 0$) and restriction to the relative admissible strains (those such that $\mathbf{E} \cdot \mathbf{V} = 0$) yield two materials in the same class, the former softer than the latter.^a

By comparing (2.12) and (2.13) with (2.3), we see that the first-gradient plate functional may be regarded as a “three-dimensional” stored-energy functional for a constrained \mathbf{e}_3 -transversely isotropic linearly elastic material in two ways: (i) by setting

$$\sigma_{(1)} = \check{\sigma}_{(1)} \quad \text{and} \quad E = 4\mu \frac{\lambda + \mu}{\lambda + 2\mu}, \quad \nu = \frac{\lambda}{\lambda + 2\mu}, \quad (2.16)$$

^aStress and strain constraints are discussed in [9].

if (2.9)₁ is taken into account as an *evaluation constraint*, according to definition (2.11)₁; (ii) by setting

$$\sigma_{(1)} = \hat{\sigma}_{(1)} \quad \text{and} \quad E = 4\mu \frac{\hat{\lambda} + \mu}{\hat{\lambda} + 2\mu}, \quad \nu = \frac{\hat{\lambda}}{\hat{\lambda} + 2\mu}, \tag{2.17}$$

if (2.9)₁ is taken into account as a *minimization constraint*, as in (2.14) combined with definition (2.11)₂; in both cases, E and ν measure, respectively, the *extensional stiffness* and the *lateral-contraction compliance* of material fibers perpendicular to \mathbf{e}_3 ; in either case, the value-wise identification (2.4) holds true. We shall prove in Sec. 5 that the functional

$$\hat{\Sigma}_{(1)}(\mathbf{u}) = \int_C \hat{\sigma}_{(1)}(\mathbf{E}(\mathbf{u})) \tag{2.18}$$

is the Γ -limit restituting the Reissner–Mindlin functional (2.1).

2.5. The second-gradient plate functional

For a full understanding of position and use of the Reissner–Mindlin plate theory with respect to three-dimensional elasticity, it remains for us to investigate the role of the second-order partial differential equations in system (2.9).

In [4], a paper where a weak-balance format to derive plate theories of Reissner–Mindlin type has been employed in the place of a less general variational format, (2.9)_{2,3} have been interpreted as standard constraints on the *hyperstrain*

$$\mathbf{G}(\mathbf{u}) = \nabla \nabla \mathbf{u} \quad (G_{ijk} = u_{i,jk}),$$

accompanied by *reaction hyperstresses* additional to the ordinary reaction stresses accompanying the strain constraint (2.9)₁; and, as to the use of plate theories of the Reissner–Mindlin type to approximate three-dimensional elasticity problems, it has been shown how such combined reaction fields can help restore equilibrium at both interior and boundary points of a plate-like cylinder comprised of a constrained \mathbf{e}_3 -transversely isotropic linearly elastic material with stored-energy density $\check{\sigma}_{(1)}$ as in (2.12).

In this paper, we associate the second-order partial differential equations (2.9)_{2,3} with the *second-gradient plate functional*

$$\Sigma_{(2)}(\mathbf{u}) = \int_C \sigma_{(2)}(\mathbf{G}(\mathbf{u})), \quad \text{where} \quad \sigma_{(2)}(\mathbf{G}) = \frac{1}{2} \tau_P \sum_{\alpha} G_{\alpha 33}^2, \quad \tau_P > 0,$$

and we define the *plate functional* Σ_P as follows:

$$\Sigma_P(\mathbf{u}) = \hat{\Sigma}_{(1)}(\mathbf{u}) + \Sigma_{(2)}(\mathbf{u}), \tag{2.19}$$

with $\hat{\Sigma}_{(1)}$ given by (2.18). Note that

$$\sigma_2(\mathbf{G}) \equiv 0 \Leftrightarrow (2.9)_{2,3} \text{ hold,}$$

and that, consequently,

$$\Sigma_P(\mathbf{u}_{RM}) = \hat{\Sigma}_{(1)}(\mathbf{u}_{RM}).$$

3. Choice of the Family of Functionals

We introduce an auxiliary parameter $\varepsilon \in (0, 1]$, and we let $\mathcal{P}(\varepsilon)$ be a plane domain as in the previous section. We decompose the boundary of $\mathcal{P}(\varepsilon)$ as follows:

$$\partial\mathcal{P}(\varepsilon) = \partial_d\mathcal{P}(\varepsilon) \cup \partial_\ell\mathcal{P}(\varepsilon), \quad \text{with} \quad \partial_d\mathcal{P}(\varepsilon) \cap \partial_\ell\mathcal{P}(\varepsilon) = \emptyset.$$

For $\mathcal{C}(\varepsilon)$, the right cylinder with cross-section $\mathcal{P}(\varepsilon)$ and height $2h(\varepsilon)$, we write

$$\partial_d\mathcal{C}(\varepsilon) = \partial_d\mathcal{P}(\varepsilon) \times (-h(\varepsilon), h(\varepsilon)), \quad \partial_\ell\mathcal{C}(\varepsilon) = \partial_\ell\mathcal{P}(\varepsilon) \times (-h(\varepsilon), h(\varepsilon)).$$

We consider a plate-like body occupying the region $\mathcal{C}(\varepsilon)$, made of a linearly elastic material with stored-energy density:

$$\sigma(\mathbf{E}(\mathbf{u}), \mathbf{G}(\mathbf{u})) = \sigma_{t-i}(\mathbf{E}(\mathbf{u})) + \sigma_{(2)}(\mathbf{G}(\mathbf{u})).$$

For simplicity, we take the body clamped on $\partial_d\mathcal{C}(\varepsilon)$ and subject to contact-load distributions \mathbf{c} on the remaining part of its boundary. Denoting by \mathbf{d} the distance loads, the *load potential* is

$$\Delta(\mathbf{u}, \varepsilon) := \int_{\mathcal{C}(\varepsilon)} \mathbf{d} \cdot \mathbf{u} + \int_{\mathcal{P}(\varepsilon)} \mathbf{c}^\pm \cdot \mathbf{u}^\pm + \int_{\partial_\ell\mathcal{C}(\varepsilon)} \mathbf{c} \cdot \mathbf{u}, \quad (3.1)$$

where \mathbf{c}^\pm and \mathbf{u}^\pm denote, respectively, the restrictions of \mathbf{c} and \mathbf{u} to the top and bottom ends of $\mathcal{C}(\varepsilon)$. For the body we consider here, equilibrium is characterized by minimization of the *total potential*:

$$\Pi(\mathbf{u}, \varepsilon) = \Sigma(\mathbf{u}, \varepsilon) - \Delta(\mathbf{u}, \varepsilon), \quad (3.2)$$

where

$$\Sigma(\mathbf{u}, \varepsilon) = \int_{\mathcal{C}(\varepsilon)} \sigma(\mathbf{E}(\mathbf{u}), \mathbf{G}(\mathbf{u}))$$

is the *elastic potential*.

We denote by (x_1, x_2, x_3) and $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ the coordinates of the typical points of $\mathcal{C}(\varepsilon)$ and $\mathcal{C} = \mathcal{C}(1)$, respectively; we also set $\mathcal{P} = \mathcal{P}(1)$, $\partial_\ell\mathcal{C} = \partial_\ell\mathcal{C}(1)$, and $\partial_d\mathcal{C} = \partial_d\mathcal{C}(1)$. Following [5, 6], we assume that $\mathcal{C}(\varepsilon)$ is mapped one-to-one into the region \mathcal{C} by the transformation:

$$x_\alpha = \varepsilon^p \bar{x}_\alpha, \quad x_3 = \varepsilon^q \bar{x}_3, \quad (3.3)$$

and we scale displacement, material moduli, and loads as follows:

$$u_\alpha = \varepsilon^m \bar{u}_\alpha, \quad u_3 = \varepsilon^n \bar{u}_3, \quad (3.4)$$

$$\bar{\lambda} = \varepsilon^{-r} \lambda, \quad \bar{\mu} = \varepsilon^{-r} \mu, \quad \bar{\tau}_3 = \varepsilon^{-u} \tau_3, \quad \bar{\tau}_1 = \varepsilon^{-v} \tau_1, \quad \bar{\tau}_2 = \varepsilon^{-z} \tau_2, \quad \bar{\tau}_P = \varepsilon^{-i} \tau_P, \quad (3.5)$$

and

$$\begin{aligned} \bar{d}_\alpha &= \varepsilon^{-s} d_\alpha, & \bar{d}_3 &= \varepsilon^{-t} d_3, \\ \bar{c}_\alpha^\pm &= \varepsilon^{-y} c_\alpha^\pm, & \bar{c}_3^\pm &= \varepsilon^{-w} c_3^\pm. \end{aligned} \quad (3.6)$$

The *scaling exponents* $(m, n, p, q; r, u, v, z, i, s, t)$ are relative integers that we choose consistently with the following restrictions: (i) $E_{3\alpha}$ and $E_{\alpha 3}$ scale the same, as well

as the work forms associated with surface and body loads; (ii) the *scaled stored-energy density*:

$$\begin{aligned} \bar{\sigma}_{t-i}(\bar{\mathbf{E}}) &:= \frac{1}{2}\bar{\tau}_1(\bar{E}_{33})^2 + \bar{\tau}_2(\bar{E}_{11} + \bar{E}_{22})\bar{E}_{33} + 2\bar{\tau}_3(\bar{E}_{13}^2 + \bar{E}_{23}^2) \\ &\quad + \frac{1}{2}(\bar{\lambda} + 2\bar{\mu})(\bar{E}_{11} + \bar{E}_{22})^2 - 4\bar{\mu}(\bar{E}_{11}\bar{E}_{22} - \bar{E}_{12}^2) \end{aligned} \quad (3.7)$$

is positive. Such requirements are met provided that the scaling exponents satisfy the following restrictions:

$$q = m - n + p, \quad r = 2z - v, \quad s + q = y, \quad t + q = w. \quad (3.8)$$

In view of (3.8), it suffices to choose $\ell = \{m, n, p; u, v, z; s, t\}$ as a list of independent exponents.

Scaling the stored-energy functionals $\Sigma(\mathbf{u}, \varepsilon)$ according to (3.3)–(3.5), yields an ε -family of *auxiliary functionals*:

$$\widehat{\Sigma}(\bar{\mathbf{u}}, \varepsilon; \hat{\alpha}, \hat{\beta}_1, \hat{\beta}_2, \hat{\gamma}, \hat{t}_1) = \varepsilon^{\hat{\alpha}}\mathbf{A}(\bar{\mathbf{u}}) + \varepsilon^{\hat{\beta}_1}\mathbf{B}_1(\bar{\mathbf{u}}) + \varepsilon^{\hat{\beta}_2}\mathbf{B}_2(\bar{\mathbf{u}}) + \varepsilon^{\hat{\gamma}}\Gamma(\bar{\mathbf{u}}) + \varepsilon^{\hat{t}_1}\mathbf{I}(\bar{\mathbf{u}}), \quad (3.9)$$

where

$$\begin{aligned} \hat{\alpha} &= -m + 3n + p + v, & \hat{\beta}_1 &= m + n + p + u, & \hat{\beta}_2 &= m + n + p + z, \\ \hat{\gamma} &= 3m - n + p + 2z - v, & \hat{t}_1 &= i + 2p - q, \end{aligned} \quad (3.10)$$

and

$$\begin{aligned} \mathbf{A}(\bar{\mathbf{u}}) &= \int_C \frac{1}{2}\bar{\tau}_1\bar{E}_{33}^2, & \mathbf{B}_1(\bar{\mathbf{u}}) &= \int_C 2\bar{\tau}_3(\bar{E}_{13}^2 + \bar{E}_{23}^2), \\ \mathbf{B}_2(\bar{\mathbf{u}}) &= \int_C \bar{\tau}_2\bar{E}_{33}(\bar{E}_{11} + \bar{E}_{22}), \\ \Gamma(\bar{\mathbf{u}}) &= \int_C \frac{1}{2}((\bar{\lambda} + 2\bar{\mu})(\bar{E}_{11} + \bar{E}_{22})^2 - 4\bar{\mu}(\bar{E}_{11}\bar{E}_{22} - \bar{E}_{12}^2)), \\ \mathbf{I}(\bar{\mathbf{u}}) &= \int_C \frac{1}{2}\bar{\tau}_P \sum_{\alpha} (\bar{u}_{\alpha,33})^2. \end{aligned} \quad (3.11)$$

It follows from (3.10) that the exponents appearing in (3.9) must satisfy

$$\hat{\alpha} - 2\hat{\beta}_2 + \hat{\gamma} = 0. \quad (3.12)$$

Using (3.3), (3.4), and (3.6), the load potentials Δ are transformed into the family

$$\widehat{\Delta}(\bar{\mathbf{u}}, \varepsilon; \hat{\delta}_1, \hat{\delta}_2) = \varepsilon^{\hat{\delta}_1}\Delta_1(\bar{\mathbf{u}}) + \varepsilon^{\hat{\delta}_2}\Delta_2(\bar{\mathbf{u}}) \quad (3.13)$$

where $\hat{\delta}_1 = m + 2p + q + s$, $\hat{\delta}_2 = n + 2p + q + t$ and

$$\begin{aligned} \Delta_1(\bar{\mathbf{u}}) &:= \int_C \sum_{\alpha} \bar{d}_{\alpha}\bar{u}_{\alpha} + \int_{\mathcal{P}} \sum_{\alpha} \bar{c}_{\alpha}^{\pm}\bar{u}_{\alpha}^{\pm} + \int_{\partial_{\ell}C} \sum_{\alpha} \bar{c}_{\alpha}\bar{u}_{\alpha}, \\ \Delta_2(\bar{\mathbf{u}}) &:= \int_C \bar{d}_3\bar{u}_3 + \int_{\mathcal{P}} \bar{c}_3^{\pm}\bar{u}_3^{\pm} + \int_{\partial_{\ell}C} \bar{c}_3\bar{u}_3. \end{aligned} \quad (3.14)$$

All choices of the scaling exponents producing the same list of *energy exponents*

$$\hat{\ell} \equiv (\hat{\alpha}, \hat{\beta}_1, \hat{\beta}_2, \hat{\gamma}, \hat{\iota}, \hat{\delta}_1, \hat{\delta}_2)$$

are equivalent; each such list determines a family of functionals

$$\hat{\Pi}(\cdot, \varepsilon; \hat{\ell}) := \hat{\Sigma}(\cdot, \varepsilon; \hat{\alpha}, \dots, \hat{\iota}_1) - \hat{\Delta}(\cdot, \varepsilon; \hat{\delta}_1, \hat{\delta}_2),$$

defined over the class $\bar{\mathcal{U}}$ of all displacements $\bar{\mathbf{u}}$ such as to keep all integrals $A(\bar{\mathbf{u}}), \dots, \Delta(\bar{\mathbf{u}})$ in (3.11) and (3.14) finite. It has been shown in [6] that, if as a selection criterion for consistent choices of exponent lists $\hat{\ell}$ and subclasses of $\bar{\mathcal{U}}$ one takes the boundedness requirement

$$\lim_{\varepsilon \rightarrow 0} \hat{\Pi}(\bar{\mathbf{u}}, \varepsilon; \hat{\ell}) < +\infty, \tag{3.15}$$

then those and only those exponent lists are selected that yield one or another of the functionals of the classic theories of thin structures, each defined over a characteristic subclass of $\bar{\mathcal{U}}$ and obtained by de-scaling the functional

$$\hat{\Pi}(\cdot, 0; \hat{\ell}) := \lim_{\varepsilon \rightarrow 0} \hat{\Pi}(\cdot, \varepsilon; \hat{\ell}).$$

In particular, it is not hard to check that, if

$$\hat{\alpha} < 0, \quad \hat{\beta}_1 = \hat{\gamma} = 0, \quad \hat{\iota} < 0, \tag{3.16}$$

then: (i) the limit in (3.15) is finite if and only if $\bar{\mathbf{u}}$ solves (2.9); (ii) de-scaling the quadratic part of $\hat{\Pi}(\cdot, 0; \hat{\ell})$ yields the first-gradient plate functional $\Sigma_{(1)}$, with the integrand as in (2.16).

Not surprisingly, the choice of scaling exponents used in most asymptotic derivations of plate theories, namely,^b

$$m = 1, \quad n = 0, \quad p = 0, \quad q = 1, \tag{3.17}$$

is compatible with (3.16). Thus, we adopt (3.17), and use (3.10) and (3.16) to determine the remaining exponents. Since

$$\hat{\alpha} = -1 + v, \quad \hat{\beta}_1 = 1 + u, \quad \hat{\beta}_2 = 1 + z, \quad \hat{\gamma} = 3 + 2z - v, \quad \hat{\iota} = i - 1, \tag{3.18}$$

conditions (3.16)_{2,3} are met if we choose

$$u = -1, \quad r = -3 = 2z - v$$

(note that $r = -3$ is another standard choice); on the other hand, (3.16)_{1,4} enforce $v < 1, i < 1$, which we comply with by taking

$$v = i = -1.$$

As to scaling the loads, we choose $t = -1, s = -2$, whence

$$\hat{\delta}_1 = \hat{\delta}_2 = 0.$$

The resulting list of energy exponents is

$$(-4, 0, -2, 0, -1, 0, 0) =: \hat{\ell}_{RM},$$

^bSee, for example, [1] or [3].

which corresponds to the family of auxiliary functionals:

$$\widehat{\Pi}_{RM}(\bar{\mathbf{u}}, \varepsilon) = \widehat{\Sigma}_{RM}(\bar{\mathbf{u}}, \varepsilon) - \widehat{\Delta}(\bar{\mathbf{u}}), \tag{3.19}$$

with

$$\widehat{\Delta}(\bar{\mathbf{u}}) = \widehat{\Delta}(\bar{\mathbf{u}}, 1),$$

and with

$$\widehat{\Sigma}_{RM}(\bar{\mathbf{u}}, \varepsilon) = \int_C (\bar{\sigma}_{(1)}^\varepsilon(\bar{\mathbf{E}}) + \bar{\sigma}_{(2)}^\varepsilon(\bar{\mathbf{G}})), \quad \bar{\mathbf{E}} = \mathbf{E}(\bar{\mathbf{u}}), \quad \bar{\mathbf{G}} = \mathbf{G}(\bar{\mathbf{u}}), \tag{3.20}$$

where

$$\begin{aligned} \bar{\sigma}_{(1)}^\varepsilon(\bar{\mathbf{E}}) &= \frac{1}{2} \bar{\tau}_1 \left(\frac{\bar{E}_{33}}{\varepsilon} \right)^2 + \bar{\tau}_2 (\bar{E}_{11} + \bar{E}_{22}) \frac{\bar{E}_{33}}{\varepsilon} + 2\bar{\tau}_3 (\bar{E}_{13}^2 + \bar{E}_{23}^2) \\ &\quad + \frac{1}{2} (\bar{\lambda} + 2\bar{\mu}) \left((\bar{E}_{11} + \bar{E}_{22})^2 - \frac{4\bar{\mu}}{\bar{\lambda} + 2\bar{\mu}} (\bar{E}_{11}\bar{E}_{22} - \bar{E}_{12}^2) \right), \end{aligned} \tag{3.21}$$

and

$$\bar{\sigma}_{(2)}^\varepsilon(\bar{\mathbf{G}}) = \frac{1}{2} \bar{\tau}_P \left(\left(\frac{\bar{G}_{133}}{\varepsilon} \right)^2 + \left(\frac{\bar{G}_{233}}{\varepsilon} \right)^2 \right). \tag{3.22}$$

This is the family of energy functionals we shall employ to validate the Reissner–Mindlin plate theory by our variational convergence argument, in Sec. 5.

4. Choice of the Γ -Limit

In this section, we restrict our attention to the ε -family $\widehat{\Sigma}_{RM}(\cdot, \varepsilon)$ of quadratic parts of the energy functionals $\widehat{\Pi}_{RM}(\cdot, \varepsilon)$. We assume that it has a Γ -limit $\widetilde{\Sigma}_{RM}$ as $\varepsilon \rightarrow 0$, and we examine this limit in detail. Our arguments are purely heuristic in that we do not take care of analytical or topological issues. In particular, we do not specify in which spaces the sequences of functions considered here are supposed to converge.

By definition [2], the family $\widehat{\Sigma}_{RM}(\cdot, \varepsilon)$ Γ -converges to the functional $\widetilde{\Sigma}_{RM}$ if:

(a) for every $\bar{\mathbf{u}}$ and every sequence $\bar{\mathbf{u}}^\varepsilon$ such that $\bar{\mathbf{u}}^\varepsilon \rightarrow \bar{\mathbf{u}}$,

$$\liminf_{\varepsilon \rightarrow 0} \widehat{\Sigma}_{RM}(\bar{\mathbf{u}}^\varepsilon, \varepsilon) \geq \widetilde{\Sigma}_{RM}(\bar{\mathbf{u}}); \tag{4.1}$$

(b) for every $\bar{\mathbf{u}}$, there is a sequence $\bar{\mathbf{u}}^\varepsilon$ such that

$$\lim_{\varepsilon \rightarrow 0} \widehat{\Sigma}_{RM}(\bar{\mathbf{u}}^\varepsilon, \varepsilon) = \widetilde{\Sigma}_{RM}(\bar{\mathbf{u}}). \tag{4.2}$$

In order to guess $\widetilde{\Sigma}_{RM}$, for each displacement field $\bar{\mathbf{u}} : \mathcal{C} \rightarrow \mathbb{R}^3$, we consider any sequence $\bar{\mathbf{u}}^\varepsilon \rightarrow \bar{\mathbf{u}}$ such that

$$\bar{\mathbf{u}}^\varepsilon = \bar{\mathbf{u}} + \varepsilon \bar{\mathbf{w}} + o(\varepsilon), \quad \text{for some vector field } \bar{\mathbf{w}}.$$

We have from (3.20)–(3.22), (2.8) and the positivity of $\bar{\tau}_P$ that, for the limit on the left-hand side of (4.1) to be finite, the following integrals must be uniformly bounded in ε :

$$\int_{\mathcal{C}} \left(\frac{\bar{u}_{3,3}^\varepsilon}{\varepsilon} \right)^2 = \int_{\mathcal{C}} \left(\frac{\bar{u}_{3,3}}{\varepsilon} + \bar{w}_{3,3} \right)^2 \quad \text{and} \quad \int_{\mathcal{C}} \left(\frac{\bar{u}_{\alpha,33}^\varepsilon}{\varepsilon} \right)^2 = \int_{\mathcal{C}} \left(\frac{\bar{u}_{\alpha,33}}{\varepsilon} + \bar{w}_{\alpha,33} \right)^2 .$$

Consequently, $\bar{\mathbf{u}}$ must solve (2.9) and, moreover,

$$\lim_{\varepsilon \rightarrow 0} \widehat{\Sigma}_{RM}(\bar{\mathbf{u}}^\varepsilon, \varepsilon) = \int_{\mathcal{C}} \left(\bar{\sigma}_{t-i}(\mathbf{E}(\bar{\mathbf{u}}) + g \mathbf{e}_3 \otimes \mathbf{e}_3) + \frac{1}{2} \bar{\tau}_P \sum_{\alpha} \psi_{\alpha}^2 \right), \quad (4.3)$$

where

$$g = \bar{w}_{3,3}, \quad \psi_{\alpha} = \bar{w}_{\alpha,33}.$$

Note that, whatever $\bar{\mathbf{w}}$, the limit in (4.3) provides us with an upper bound for $\widehat{\Sigma}_{RM}(\bar{\mathbf{u}})$. Minimizing the right-hand side of (4.3) with respect to g and ψ_{α} , we obtain:

$$\widetilde{\Sigma}_{RM}(\bar{\mathbf{u}}) := \int_{\mathcal{C}} \tilde{\sigma}(\mathbf{E}(\bar{\mathbf{u}})), \quad \tilde{\sigma}(\bar{\mathbf{E}}) := \min\{\bar{\sigma}_{t-i}(\bar{\mathbf{E}} + g\bar{\mathbf{V}}) : g \in \mathbb{R}\} \quad (4.4)$$

as a *candidate* Γ -limit for the family $\widehat{\Sigma}_{RM}(\cdot, \varepsilon)$. Note that, by repeating the computation performed to evaluate $\hat{\sigma}_{(1)}$, one finds that

$$\tilde{\sigma}(\bar{\mathbf{E}}) = 2\bar{\tau}_3(\bar{E}_{13}^2 + \bar{E}_{23}^2) + \frac{1}{2}(\tilde{\lambda} + 2\bar{\mu}) \left((\bar{E}_{11} + \bar{E}_{22})^2 - \frac{4\bar{\mu}}{\tilde{\lambda} + 2\bar{\mu}}(\bar{E}_{11}\bar{E}_{22} - \bar{E}_{12}^2) \right),$$

where $\tilde{\lambda} = \bar{\lambda} - \bar{\tau}_2^2/\bar{\tau}_1$.

The second requirement in the definition of Γ -convergence is that, given any scaled Reissner–Mindlin displacement field $\bar{\mathbf{u}}_{RM}$, one can exhibit an associated recovery sequence $\bar{\mathbf{u}}_{RM}^\varepsilon$, that is, a sequence for which

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathcal{C}} (\bar{\sigma}_{(1)}^\varepsilon(\mathbf{E}(\bar{\mathbf{u}}_{RM}^\varepsilon)) + \bar{\sigma}_{(2)}^\varepsilon(\mathbf{G}(\bar{\mathbf{u}}_{RM}^\varepsilon))) = \int_{\mathcal{C}} \tilde{\sigma}(\mathbf{E}(\bar{\mathbf{u}}_{RM})).$$

With a view to choosing such a recovery sequence, note that

$$\tilde{\sigma}(\bar{\mathbf{E}}) = \min\{\bar{\sigma}_{(1)}^\varepsilon(\bar{\mathbf{E}} + g\bar{\mathbf{V}}) : g \in \mathbb{R}\},$$

and that definition (4.4)₂ is nothing but the rescaled version of the right-hand side of the (2.14). Hence,

$$\tilde{\sigma} = \bar{\sigma}_{t-i}|_{\mathbf{S} \cdot \mathbf{V}=0} = \bar{\sigma}_{(1)}^\varepsilon|_{\mathbf{S}^\varepsilon \cdot \mathbf{V}=0}, \quad (4.5)$$

where \mathbf{S}^ε denotes the derivative of $\bar{\sigma}_{(1)}^\varepsilon$. The stress constraint $\mathbf{S} \cdot \mathbf{V} = 0$ entering (4.5) is equivalent to the strain constraint

$$\mathbf{E} \cdot \bar{\mathbf{V}} = 0, \quad \bar{\mathbf{V}} = \tau_1 \mathbf{e}_3 \otimes \mathbf{e}_3 + \tau_2(\mathbf{1} - \mathbf{e}_3 \otimes \mathbf{e}_3), \quad (4.6)$$

which is nonstandard [9], in that the implied class of admissible strains depends on the material ratio τ_2/τ_1 :

$$E_{33} = -\frac{\tau_2}{\tau_1}(E_{11} + E_{22}). \tag{4.7}$$

Such a workingless strain, if expressed in terms of the displacement $\bar{\mathbf{u}}_{RM}$, leads to the relation

$$(\bar{\mathbf{u}}_{RM})_{3,3} = -\frac{\tau_2}{\tau_1} \sum_{\alpha} (\bar{\mathbf{u}}_{RM})_{\alpha,\alpha},$$

whence, on taking (2.9)_{2,3} into account and using the notation adopted in (2.5), we obtain the following *augmented Reissner–Mindlin displacement field*:

$$\bar{\mathbf{u}}_{RM}^{(a)} = \bar{\mathbf{u}}_{RM} - \frac{\tau_2}{2\tau_1} \bar{x}_3^2 \sum_{\alpha} \bar{\varphi}_{\alpha,\alpha} \mathbf{e}_3. \tag{4.8}$$

Similarly, the stress constraint $\mathbf{S}^\varepsilon \cdot \mathbf{V} = 0$ appearing in (4.5) leads to

$$\bar{\mathbf{u}}_{RM}^\varepsilon := \bar{\mathbf{u}}_{RM} - \varepsilon \bar{a} \mathbf{e}_3, \quad \bar{a} := \frac{\tau_2}{2\tau_1} \bar{x}_3^2 \sum_{\alpha} \bar{\varphi}_{\alpha,\alpha}. \tag{4.9}$$

A simple computation then shows that $\bar{\mathbf{u}}_{RM}^\varepsilon \rightarrow \bar{\mathbf{u}}_{RM}$ as $\varepsilon \rightarrow 0$; that

$$\lim_{\varepsilon \rightarrow 0} \bar{\sigma}_{(1)}^\varepsilon(\mathbf{E}(\bar{\mathbf{u}}_{RM}^\varepsilon)) = \bar{\sigma}(\mathbf{E}(\bar{\mathbf{u}}_{RM}));$$

and that $\{\bar{\mathbf{u}}_{RM}^\varepsilon\}$ is a recovery sequence.

5. Variational Convergence

Throughout this section, we denote by C a positive constant which may vary from one formula to another, and we omit relabeling sequences. Also, we keep calling “sequence” any family of functions labeled by a continuous parameter ε .

Let $\bar{\mathbf{d}} \in L^2(\mathcal{C}, \mathbb{R}^3)$, $\bar{\mathbf{c}} \in L^2(\partial_d \mathcal{C}, \mathbb{R}^3)$, and $\bar{\mathbf{c}}^\pm \in L^2(\mathcal{P}, \mathbb{R}^3)$. The family of functionals $\widehat{\Pi}_{RM}(\cdot, \varepsilon)$ in (3.19) is well-defined over the *space of admissible displacements*

$$\mathcal{A} := \{\bar{\mathbf{u}} \in H^1(\mathcal{C}, \mathbb{R}^3) \mid \bar{u}_{\alpha,33} \in L^2(\mathcal{C}), \bar{\mathbf{u}} = \mathbf{0} \text{ on } \partial_d \mathcal{P} \times (-h, h)\}.$$

For reasons that will be apparent later, we find it convenient to extend the domain of definition of the functionals $\widehat{\Pi}_{RM}$ to all of $L^2(\mathcal{C}, \mathbb{R}^3)$, by setting

$$\widehat{\Pi}_{RM}^\varepsilon(\bar{\mathbf{u}}, \varepsilon) := \begin{cases} \widehat{\Pi}_{RM}(\bar{\mathbf{u}}, \varepsilon) & \text{if } \bar{\mathbf{u}} \in \mathcal{A}, \\ +\infty & \text{otherwise.} \end{cases}$$

We also introduce the class of *Reissner–Mindlin displacements*:

$$\mathcal{RM} := \{\bar{\mathbf{u}} \in H^1(\mathcal{C}, \mathbb{R}^3) \mid \exists \hat{\varphi}_\alpha, \hat{u}_3 \in H_{0d}^1(\mathcal{P}) : \bar{u}_\alpha = \bar{x}_3 \hat{\varphi}_\alpha(\bar{x}_1, \bar{x}_2), \bar{u}_3 = \hat{u}_3(\bar{x}_1, \bar{x}_2)\},$$

where

$$H_{0d}^1(\mathcal{P}) = \{\xi \in H^1(\mathcal{P}) \mid \xi = 0 \text{ on } \partial_d \mathcal{P}\}.$$

The class \mathcal{RM} is brought to attention by the following result, that makes precise in which sense, under the assumptions (3.16), the boundedness requirement (3.15) delivers the constraint (2.9) as $\varepsilon \rightarrow 0$.

Lemma 5.1. *Let $\{\bar{\mathbf{u}}^\varepsilon\} \subset \mathcal{A}$ be a sequence such that*

$$\sup_\varepsilon \widehat{\Pi}_{RM}(\bar{\mathbf{u}}^\varepsilon, \varepsilon) < +\infty. \tag{5.1}$$

Then, there exists a subsequence of $\{\bar{\mathbf{u}}^\varepsilon\}$ and an element $\bar{\mathbf{u}} \in \mathcal{RM}$ such that

$$\bar{\mathbf{u}}^\varepsilon \rightharpoonup \bar{\mathbf{u}} \text{ in } H^1(\mathcal{C}, \mathbb{R}^3).$$

Proof. Given any sequence satisfying (5.1), it follows from (3.20)–(3.22), (2.8), and the positivity of $\bar{\tau}_P$, that

$$\widehat{\Sigma}_{RM}(\bar{\mathbf{u}}^\varepsilon, \varepsilon) \geq C \int_{\mathcal{C}} \left(\sum_{i,\alpha} E_{i\alpha}(\bar{\mathbf{u}}^\varepsilon) + \left(\frac{\bar{u}_{3,3}^\varepsilon}{\varepsilon}\right)^2 + \sum_\alpha \left(\frac{\bar{u}_{\alpha,33}^\varepsilon}{\varepsilon}\right)^2 \right).$$

Furthermore, by Young’s inequality and the continuity of the trace operator in H^1 ,

$$2|\Delta(\bar{\mathbf{u}}^\varepsilon, \varepsilon)| \leq \frac{1}{\alpha} \left(\int_{\mathcal{C}} |\bar{\mathbf{d}}|^2 + \int_{\partial_\ell \mathcal{C}} |\bar{\mathbf{c}}|^2 + \int_{\mathcal{P}} |\bar{\mathbf{c}}^\pm|^2 \right) + \alpha \|\bar{\mathbf{u}}^\varepsilon\|_{H^1}^2,$$

for all positive α . A combination of the above estimates with Korn’s inequality yields:

$$\widehat{\Pi}_{RM}(\bar{\mathbf{u}}^\varepsilon, \varepsilon) \geq C \left(\int_{\mathcal{C}} \left(\left(\frac{\bar{u}_{3,3}^\varepsilon}{\varepsilon}\right)^2 + \sum_\alpha \left(\frac{\bar{u}_{\alpha,33}^\varepsilon}{\varepsilon}\right)^2 \right) + \|\bar{\mathbf{u}}^\varepsilon\|_{H^1}^2 - 1 \right),$$

for α sufficiently small. Hence, all terms on the right are uniformly bounded. Passing to a subsequence, we obtain the desired result:

$$\bar{\mathbf{u}}^\varepsilon \rightharpoonup \bar{\mathbf{u}} \text{ in } H^1(\mathcal{C}, \mathbb{R}^3), \quad \text{with } \bar{u}_{3,3} = 0 \quad \text{and} \quad \bar{u}_{\alpha,33} = 0. \quad \square$$

Remark 5.2. Using the estimates obtained in the last proof, one can show by the direct methods of the Calculus of Variations that the strictly convex functional $\widehat{\Pi}_{RM}(\cdot, \varepsilon)$ has a unique minimizer.

The functional

$$\widetilde{\Pi}_{RM} := \widetilde{\Sigma}_{RM} - \widehat{\Delta} \tag{5.2}$$

is well-defined on \mathcal{RM} . We extend it to the whole of $L^2(\mathcal{C}, \mathbb{R}^3)$ by setting

$$\widetilde{\Pi}_{RM}^e(\bar{\mathbf{u}}) := \begin{cases} \widetilde{\Pi}_{RM}(\bar{\mathbf{u}}) & \text{if } \bar{\mathbf{u}} \in \mathcal{A}, \\ +\infty & \text{otherwise.} \end{cases}$$

Needless to say, minimizing $\widehat{\Pi}_{RM}$ and $\widetilde{\Pi}_{RM}$ is equivalent to minimizing the corresponding extended functionals.

Theorem 5.3. *The family $\widehat{\Pi}_{RM}^e(\cdot, \varepsilon)$ sequentially Γ -converges to $\widetilde{\Pi}_{RM}^e$ with respect to the $L^2(\mathcal{C}, \mathbb{R}^3)$ topology, in the following sense:*

- (1) (**Liminf Inequality**) *for every sequence of positive numbers ε converging to 0, and for every pair of a sequence $\{\bar{\mathbf{u}}^\varepsilon\} \subset L^2(\mathcal{C}, \mathbb{R}^3)$ and a field $\bar{\mathbf{u}} \in L^2(\mathcal{C}, \mathbb{R}^3)$ such that $\bar{\mathbf{u}}^\varepsilon \rightarrow \bar{\mathbf{u}}$ in $L^2(\mathcal{C}, \mathbb{R}^3)$,*

$$\liminf_{\varepsilon \rightarrow 0} \widehat{\Pi}_{RM}^e(\bar{\mathbf{u}}^\varepsilon, \varepsilon) \geq \widetilde{\Pi}_{RM}^e(\bar{\mathbf{u}});$$

- (2) (**Recovery Sequence**) *for every $\bar{\mathbf{u}} \in L^2(\mathcal{C}, \mathbb{R}^3)$, there is a sequence $\{\bar{\mathbf{u}}^\varepsilon\} \subset L^2(\mathcal{C}, \mathbb{R}^3)$ such that $\bar{\mathbf{u}}^\varepsilon \rightarrow \bar{\mathbf{u}}$ in $L^2(\mathcal{C}, \mathbb{R}^3)$, and*

$$\limsup_{\varepsilon \rightarrow 0} \widehat{\Pi}_{RM}^e(\bar{\mathbf{u}}^\varepsilon, \varepsilon) \leq \widetilde{\Pi}_{RM}^e(\bar{\mathbf{u}}).$$

Proof. To verify the liminf-inequality condition, let $\bar{\mathbf{u}} \in L^2(\mathcal{C}, \mathbb{R}^3)$ and $\{\bar{\mathbf{u}}^\varepsilon\} \subset L^2(\mathcal{C}, \mathbb{R}^3)$ be such that $\bar{\mathbf{u}}^\varepsilon \rightarrow \bar{\mathbf{u}}$ in $L^2(\mathcal{C}, \mathbb{R}^3)$. We assume (otherwise there is nothing to prove) that

$$\liminf_{\varepsilon \rightarrow 0} \widehat{\Pi}_{RM}^e(\bar{\mathbf{u}}^\varepsilon) < +\infty;$$

and (by passing to a subsequence, if needed) that

$$\liminf_{\varepsilon \rightarrow 0} \widehat{\Pi}_{RM}^e(\bar{\mathbf{u}}^\varepsilon) = \lim_{\varepsilon \rightarrow 0} \widehat{\Pi}_{RM}^e(\bar{\mathbf{u}}^\varepsilon).$$

From Lemma 5.1, we deduce that $\bar{\mathbf{u}} \in \mathcal{RM}$ and that there is a subsequence of $\{\mathbf{u}^\varepsilon\}$ such that

$$\bar{\mathbf{u}}^\varepsilon \rightharpoonup \bar{\mathbf{u}} \quad \text{in } H^1(\mathcal{C}, \mathbb{R}^3).$$

By (3.20)–(3.22), (4.4)₁, and (4.5),

$$\liminf_{\varepsilon \rightarrow 0} \widehat{\Sigma}_{RM}^e(\bar{\mathbf{u}}^\varepsilon, \varepsilon) \geq \liminf_{\varepsilon \rightarrow 0} \int_{\mathcal{C}} \bar{\sigma}_{(1)}^\varepsilon(\mathbf{E}(\bar{\mathbf{u}}^\varepsilon)) \geq \liminf_{\varepsilon \rightarrow 0} \int_{\mathcal{C}} \bar{\sigma}(\mathbf{E}(\bar{\mathbf{u}}^\varepsilon)) \geq \int_{\mathcal{C}} \bar{\sigma}(\mathbf{E}(\bar{\mathbf{u}}))$$

(in the last inequality, we have used the fact that convexity implies H^1 -weakly sequential lower semicontinuity). Therefore, in view of the linearity of $\widehat{\Delta}$, we have that

$$\liminf_{\varepsilon \rightarrow 0} \widehat{\Pi}_{RM}^e(\bar{\mathbf{u}}^\varepsilon, \varepsilon) = \liminf_{\varepsilon \rightarrow 0} (\widehat{\Sigma}_{RM}^e(\bar{\mathbf{u}}^\varepsilon, \varepsilon) + \widehat{\Delta}(\bar{\mathbf{u}}^\varepsilon)) = \widetilde{\Sigma}_{RM}(\bar{\mathbf{u}}) + \widehat{\Delta}(\bar{\mathbf{u}}) = \widetilde{\Pi}_{RM}^e(\bar{\mathbf{u}}).$$

To verify the recovery-sequence condition, we first consider $\bar{\mathbf{u}} \in \mathcal{RM} \cap C^\infty(\mathcal{C}, \mathbb{R}^3)$, and we show that the sequence in (4.9) — which, to lighten our notation,

we shall henceforth denote by $\{\bar{\mathbf{u}}^\varepsilon\}$ — is indeed a recovery sequence for $\bar{\mathbf{u}}$. Note that $\bar{u}_{\alpha,33}^\varepsilon = 0$, and that

$$E_{33}(\bar{\mathbf{u}}^\varepsilon) = -\varepsilon \frac{\bar{\tau}_2}{\bar{\tau}_1} \sum_{\alpha} E_{\alpha\alpha}(\bar{\mathbf{u}}^\varepsilon), \quad E_{\alpha 3}(\bar{\mathbf{u}}^\varepsilon) = E_{\alpha 3}(\bar{\mathbf{u}}) - \varepsilon \bar{a}_{,\alpha}; \quad (5.3)$$

therefore,

$$\widehat{\Sigma}_{RM}(\bar{\mathbf{u}}^\varepsilon, \varepsilon) = \widetilde{\Sigma}_{RM}(\bar{\mathbf{u}}^\varepsilon) - 2\varepsilon \int_{\mathcal{C}} 2\bar{\tau}_3 \sum_{\alpha} E_{\alpha 3}(\bar{\mathbf{u}}) \bar{a}_{,\alpha} + \varepsilon^2 \int_{\mathcal{C}} 2\bar{\tau}_3 (\bar{a}_{,1}^2 + \bar{a}_{,2}^2).$$

Moreover, since $\tilde{\sigma}$ is a quadratic form, we have that

$$\begin{aligned} \tilde{\sigma}(\mathbf{E}(\bar{\mathbf{u}}^\varepsilon)) - \tilde{\sigma}(\mathbf{E}(\bar{\mathbf{u}})) &= \sum_{i,j,k,l} \tilde{C}_{ijkl} E_{ij}(\bar{\mathbf{u}}) E_{kl}(\bar{\mathbf{u}}^\varepsilon - \bar{\mathbf{u}}) \\ &\quad + \frac{1}{2} \sum_{i,j,k,l} \tilde{C}_{ijkl} E_{ij}(\bar{\mathbf{u}}^\varepsilon - \bar{\mathbf{u}}) E_{kl}(\bar{\mathbf{u}}^\varepsilon - \bar{\mathbf{u}}), \end{aligned} \quad (5.4)$$

where \tilde{C}_{ijkl} are the components of the elasticity tensor associated with $\tilde{\sigma}$. Consequently,

$$|\widetilde{\Sigma}_{RM}(\bar{\mathbf{u}}^\varepsilon) - \widetilde{\Sigma}_{RM}(\bar{\mathbf{u}})| \leq C(\bar{\mathbf{u}}) \|\bar{\mathbf{u}}^\varepsilon - \bar{\mathbf{u}}\|_{H^1},$$

with $C(\bar{\mathbf{u}})$ a constant that depends on $\bar{\mathbf{u}}$. Since $\bar{\mathbf{u}}^\varepsilon \rightarrow \bar{\mathbf{u}}$ in $H^1(\mathcal{C}, \mathbb{R}^3)$, we deduce that

$$\lim_{\varepsilon \rightarrow 0} \widehat{\Sigma}(\bar{\mathbf{u}}^\varepsilon, \varepsilon) = \widetilde{\Sigma}_{RM}(\bar{\mathbf{u}}),$$

whence, by the linearity of $\widehat{\Delta}$,

$$\lim_{\varepsilon \rightarrow 0} \widehat{\Pi}_{RM}^e(\bar{\mathbf{u}}^\varepsilon, \varepsilon) = \widetilde{\Pi}_{RM}^e(\bar{\mathbf{u}}).$$

Next, for $\bar{\mathbf{u}} \in \mathcal{RM}$ arbitrary, we let $\{\bar{\mathbf{u}}_k\}_{k \in \mathbb{N}} \subset \mathcal{RM} \cap \mathcal{C}^\infty(\mathcal{C}, \mathbb{R}^3)$ be a sequence such that $\bar{\mathbf{u}}_k \rightarrow \bar{\mathbf{u}}$ in $H^1(\mathcal{C}, \mathbb{R}^3)$ as $k \rightarrow \infty$, so that, by bounded convergence,

$$\lim_{k \rightarrow \infty} \widetilde{\Pi}_{RM}^e(\bar{\mathbf{u}}_k) = \widetilde{\Pi}_{RM}^e(\bar{\mathbf{u}}).$$

For each k , let $\{\bar{\mathbf{v}}_k^\varepsilon\}$ be a recovery sequence for $\bar{\mathbf{u}}_k$. Then,

$$\lim_{k \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \widetilde{\Pi}_{RM}(\bar{\mathbf{v}}_k^\varepsilon, \varepsilon) = \lim_{k \rightarrow \infty} \widetilde{\Pi}_{RM}(\bar{\mathbf{u}}_k) = \widetilde{\Pi}_{RM}(\bar{\mathbf{u}}).$$

We conclude the proof by extracting from $\{\bar{\mathbf{v}}_k^\varepsilon\}$ a diagonal subsequence. □

Within the theory of Γ -convergence [2], the existence of a minimizer for $\widetilde{\Pi}_{RM}$, although attainable by standard methods, is a corollary of Theorem 5.3. The next theorem collects some properties of the minimizers of $\widehat{\Pi}_{RM}$.

Theorem 5.4. *Let $\bar{\mathbf{u}}_{\min}^\varepsilon$ be the unique minimizer of $\widehat{\Pi}_{RM}(\cdot, \varepsilon)$, for every $\varepsilon \in (0, 1)$, and let $\bar{\mathbf{u}}_{\min}$ be the unique minimizer of $\widetilde{\Pi}_{RM}$. Then:*

- (1) *the sequence $\widehat{\Pi}_{RM}^e(\bar{\mathbf{u}}_{\min}^\varepsilon, \varepsilon)$ converges to $\widetilde{\Pi}_{RM}(\bar{\mathbf{u}}_{\min})$;*
- (2) *the sequence $\bar{\mathbf{u}}_{\min}^\varepsilon$ converges to $\bar{\mathbf{u}}_{\min}$ strongly in $H^1(\mathcal{C}, \mathbb{R}^3)$.*

Proof. Let $\bar{\mathbf{u}} \in L^2(\mathcal{C}, \mathbb{R}^3)$. By the recovery-sequence condition, there is a sequence $\{\bar{\mathbf{u}}^\varepsilon\}$ such that $\bar{\mathbf{u}}^\varepsilon \rightarrow \bar{\mathbf{u}}$ in $L^2(\mathcal{C}, \mathbb{R}^3)$, and that

$$\tilde{\Pi}_{RM}^\varepsilon(\bar{\mathbf{u}}) \geq \limsup_{\varepsilon \rightarrow 0} \hat{\Pi}_{RM}^\varepsilon(\bar{\mathbf{u}}^\varepsilon) \geq \limsup_{\varepsilon \rightarrow 0} \hat{\Pi}_{RM}^\varepsilon(\bar{\mathbf{u}}_{\min}^\varepsilon).$$

By the compactness result in Lemma 5.1, there exists a subsequence $\bar{\mathbf{u}}_{\min}^\varepsilon$ converging weakly to some $\bar{\mathbf{u}}_{\min} \in H^1(\mathcal{C}, \mathbb{R}^3) \cap \mathcal{RM}$. By the liminf-inequality and the last chain of inequalities, we find that

$$\tilde{\Pi}_{RM}^\varepsilon(\bar{\mathbf{u}}) \geq \limsup_{\varepsilon \rightarrow 0} \hat{\Pi}_{RM}^\varepsilon(\bar{\mathbf{u}}_{\min}^\varepsilon) \geq \liminf_{\varepsilon \rightarrow 0} \hat{\Pi}_{RM}^\varepsilon(\bar{\mathbf{u}}_{\min}^\varepsilon, \varepsilon) \geq \tilde{\Pi}_{RM}^\varepsilon(\bar{\mathbf{u}}_{\min}). \quad (5.5)$$

Since $\bar{\mathbf{u}}$ was arbitrarily chosen, we deduce that $\bar{\mathbf{u}}_{\min}$ is equal to the unique minimizer $\bar{\mathbf{u}}_{\min}$. Thus, $\bar{\mathbf{u}}_{\min}^\varepsilon$ weakly converges in $H^1(\mathcal{C}, \mathbb{R}^3)$ to $\bar{\mathbf{u}}_{\min}$. Also, taking $\bar{\mathbf{u}} = \bar{\mathbf{u}}_{\min}$ in (5.5), we establish Claim 1. It remains for us to show that convergence of minimizers is strong in $H^1(\mathcal{C}, \mathbb{R}^3)$.

As in (5.4), we have that

$$\begin{aligned} \hat{\Sigma}_{RM}(\bar{\mathbf{u}}_{\min}^\varepsilon, \varepsilon) - \tilde{\Sigma}_{RM}(\bar{\mathbf{u}}_{\min}) &\geq \int_{\mathcal{C}} \tilde{\sigma}(\mathbf{E}(\bar{\mathbf{u}}_{\min}^\varepsilon)) - \tilde{\sigma}(\mathbf{E}(\bar{\mathbf{u}}_{\min})) \\ &= \int_{\mathcal{C}} \sum_{i,j,k,l} \tilde{C}_{ijkl} E_{ij}(\bar{\mathbf{u}}) E_{kl}(\bar{\mathbf{u}}_{\min}^\varepsilon - \bar{\mathbf{u}}_{\min}) \\ &\quad + \int_{\mathcal{C}} \tilde{\sigma}(\mathbf{E}(\bar{\mathbf{u}}_{\min}^\varepsilon) - \mathbf{E}(\bar{\mathbf{u}}_{\min})). \end{aligned}$$

Since

$$\tilde{\sigma}(\mathbf{A}) \geq C \sum_{\alpha,i} A_{\alpha i}^2 \quad \text{for every symmetric } \mathbf{A} \in \mathbb{R}^{3 \times 3},$$

by the continuity of the load potential and the convergence of the minima of $\hat{\Pi}_{RM}^\varepsilon$, we have that

$$0 = \limsup_{\varepsilon \rightarrow 0} \hat{\Pi}_{RM}^\varepsilon(\bar{\mathbf{u}}_{\min}^\varepsilon, \varepsilon) - \tilde{\Pi}_{RM}(\bar{\mathbf{u}}_{\min}) \geq C \limsup_{\varepsilon \rightarrow 0} \int_{\mathcal{C}} \sum_{\alpha,i} |E_{\alpha i}(\bar{\mathbf{u}}_{\min}^\varepsilon - \bar{\mathbf{u}}_{\min})|^2.$$

Moreover, as in the proof of Lemma 5.1, the boundedness of the sequence of minima guarantees that

$$\mathbf{E}_{33}(\bar{\mathbf{u}}_{\min}^\varepsilon) \rightarrow 0 = \mathbf{E}_{33}(\bar{\mathbf{u}}_{\min}) \quad \text{in } L^2(\mathcal{C}, \mathbb{R}^3).$$

Thus, we have that

$$\mathbf{E}(\bar{\mathbf{u}}_{\min}^\varepsilon - \bar{\mathbf{u}}_{\min}) \rightarrow 0 \quad \text{in } L^2(\mathcal{C}, \mathbb{R}^{3 \times 3}).$$

An application of Korn's inequality concludes the proof. □

6. Conclusions

Our mathematical justification of the two-dimensional Reissner–Mindlin plate theory within the framework of three-dimensional linear elasticity consists of several steps, which we now recapitulate.

Firstly, we choose the ε -family of energy functionals

$$\widehat{\Pi}_{RM}(\bar{\mathbf{u}}, \varepsilon) = \widehat{\Sigma}_{RM}(\bar{\mathbf{u}}, \varepsilon) - \widehat{\Delta}(\bar{\mathbf{u}})$$

defined in (3.19), where

$$\widehat{\Sigma}_{RM}(\bar{\mathbf{u}}, \varepsilon) = \int_{\mathcal{C}} (\bar{\sigma}_{(1)}^\varepsilon(\bar{\mathbf{E}}) + \bar{\sigma}_{(2)}^\varepsilon(\bar{\mathbf{G}})), \quad \bar{\mathbf{E}} = \mathbf{E}(\bar{\mathbf{u}}), \quad \bar{\mathbf{G}} = \mathbf{G}(\bar{\mathbf{u}}),$$

with

$$\begin{aligned} \bar{\sigma}_{(1)}^\varepsilon(\bar{\mathbf{E}}) &= \frac{1}{2} \bar{\tau}_1 \left(\frac{\bar{E}_{33}}{\varepsilon} \right)^2 + \bar{\tau}_2 (\bar{E}_{11} + \bar{E}_{22}) \frac{\bar{E}_{33}}{\varepsilon} + 2\bar{\tau}_3 (\bar{E}_{13}^2 + \bar{E}_{23}^2) \\ &\quad + \frac{1}{2} (\bar{\lambda} + 2\bar{\mu}) \left((\bar{E}_{11} + \bar{E}_{22})^2 - \frac{4\bar{\mu}}{\bar{\lambda} + 2\bar{\mu}} (\bar{E}_{11}\bar{E}_{22} - \bar{E}_{12}^2) \right), \end{aligned}$$

and

$$\bar{\sigma}_{(2)}^\varepsilon(\bar{\mathbf{G}}) = \frac{1}{2} \bar{\tau}_P \left(\left(\frac{\bar{G}_{133}}{\varepsilon} \right)^2 + \left(\frac{\bar{G}_{233}}{\varepsilon} \right)^2 \right).$$

Here, overbars denote scaled quantities that do not depend on the bookkeeping parameter ε ; the fields $\bar{\mathbf{u}}$, $\bar{\mathbf{E}}$, and $\bar{\mathbf{G}}$, are defined over the right cylinder \mathcal{C} of cross-section \mathcal{P} and height $2h$.

Secondly, as a candidate Γ -limit for this family, we choose the functional

$$\widetilde{\Pi}_{RM} := \widetilde{\Sigma}_{RM} - \widehat{\Delta},$$

defined in (5.2), where

$$\tilde{\sigma}(\bar{\mathbf{E}}) = 2\bar{\tau}_3 (\bar{E}_{13}^2 + \bar{E}_{23}^2) + \frac{1}{2} (\bar{\lambda} + 2\bar{\mu}) \left((\bar{E}_{11} + \bar{E}_{22})^2 - \frac{4\bar{\mu}}{\bar{\lambda} + 2\bar{\mu}} (\bar{E}_{11}\bar{E}_{22} - \bar{E}_{12}^2) \right),$$

with $\bar{\lambda} = \bar{\lambda} - \bar{\tau}_2^2/\bar{\tau}_1$.

Thirdly, we show that:

- (i) The family of functionals $\widehat{\Pi}_{RM}(\cdot, \varepsilon)$ sequentially Γ -converges to the functional $\widetilde{\Pi}_{RM}$ with respect to the $L^2(\mathcal{C}, \mathbb{R}^3)$ strong topology.
- (ii) The sequence of unique minimizers of $\widehat{\Sigma}_{RM}(\cdot, \varepsilon)$ converges in $H^1(\mathcal{C}, \mathbb{R}^3)$ to the unique minimizer of $\widetilde{\Sigma}_{RM}$, a displacement field over \mathcal{C} having the Reissner–Mindlin form

$$\bar{\mathbf{u}}_{RM}(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \bar{w}(\bar{x}_1, \bar{x}_2) \mathbf{e}_3 + \bar{x}_3 \bar{\boldsymbol{\varphi}}(\bar{x}_1, \bar{x}_2), \quad \bar{\boldsymbol{\varphi}} \cdot \mathbf{e}_3 = 0. \quad (6.1)$$

In view of (3.3) and (3.4), a displacement field $\bar{\mathbf{u}}$ over \mathcal{C} is de-scaled into

$$\mathbf{u} = \varepsilon \bar{u}_\alpha(x_1, x_2, \varepsilon^{-1}x_3) \mathbf{e}_\alpha + \bar{u}_3(x_1, x_2, \varepsilon^{-1}x_3) \mathbf{e}_3$$

over the right cylinder $\mathcal{C}(\varepsilon)$ of cross-section \mathcal{P} and height $\varepsilon(2h)$; in particular, the Reissner–Mindlin field (6.1) is de-scaled into

$$\mathbf{u}_{RM} = x_3 \boldsymbol{\varphi}(x_1, x_2) + w(x_1, x_2) \mathbf{e}_3$$

(here, $\bar{\varphi} = \varphi$ and $\bar{w} = w$). Moreover, with the use of (3.5) in addition to (3.3) and (3.4), it can be shown that the functional $\tilde{\Sigma}_{RM}^\varepsilon$ is de-scaled as follows:

$$\tilde{\Sigma}_{RM}^\varepsilon(\mathbf{u}) := \int_{C(\varepsilon)} \hat{\sigma}_{(1)}(\mathbf{E}(\mathbf{u})). \tag{6.2}$$

We have that:

- (iii) The de-scaled elastic potential takes the same values of the Reissner–Mindlin flexure-energy functional introduced in Sec. 2.1, in the following precise sense:

$$\tilde{\Sigma}_{RM}^\varepsilon(\mathbf{u}_{RM}) = \Sigma_{RM}^\varepsilon(w, \varphi), \tag{6.3}$$

where (cf. (2.1))

$$\Sigma_{RM}^\varepsilon(w, \varphi) = \frac{1}{2} \int_{\mathcal{P}} (S(\varepsilon)|\nabla w + \varphi|^2 + B(\varepsilon)((1 - \nu)|\mathbf{E}(\varphi)|^2 + \nu(\operatorname{div} \varphi)^2)),$$

with

$$S(\varepsilon) = \varepsilon 2h\tau_3, \quad B(\varepsilon) = \varepsilon^3 \frac{2}{3} h^3 \frac{E}{(1 - \nu^2)}, \quad E = 4\mu \frac{\hat{\lambda} + \mu}{\lambda + 3\mu}, \quad \nu = \frac{\hat{\lambda}}{\hat{\lambda} + 2\mu}.$$

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References

- [1] P. G. Ciarlet, *Mathematical Elasticity, Vol. II: Theory of Shells* (North-Holland Publishing Co., Amsterdam, 2000).
- [2] G. Dal Maso, *An Introduction to Γ -Convergence* (Birkhäuser, 1993).
- [3] H. Le Dret and A. Raoult, The nonlinear membrane model as variational limit of nonlinear three-dimensional elasticity, *J. Math. Pures Appl.* **74** (1995) 549–578.
- [4] M. Lembo and Podio-Guidugli, How to use reactive stresses to improve plate-theory approximations of the stress field in a linearly elastic plate-like body, *Int. J. Solids Struct.* **44**(5) (2006) 1337–1369.
- [5] B. Miara and P. Podio-Guidugli, Une approche formelle unifiée des théories de plaques et poutres linéairement élastiques, *C. R. Acad. Sci. Paris, Ser. I* **343** (2006) 675–678.
- [6] B. Miara and P. Podio-Guidugli, Deduction by scaling: A unified approach to classic plate and rod theories, to appear in *Asymptot. Anal.* (2006).
- [7] R. Paroni, P. Podio-Guidugli and G. Tomassetti, The Reissner–Mindlin plate theory via Γ -convergence, *C. R. Acad. Sci. Paris Ser. I* **343** (2006) 437–440.
- [8] P. Podio-Guidugli, On structure thinness, mechanical and variational, in *Variational Formulations in Mechanics, Theory and Applications*, eds. E. Taroco, E. A. de Souza Neto and A. A. Novotny (CIMNE, Barcelona, Spain, 2006), in press.
- [9] P. Podio-Guidugli, *Strain Constraints and Stress Constraints, A Unified Virtual-Working Format*, forthcoming (2007).

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