

DYNAMICS OF CHARGED ELASTIC BODIES UNDER DIFFUSION AT LARGE STRAINS

TOMÁŠ ROUBÍČEK*

Mathematical Institute, Charles University
Sokolovská 83, 18675 Praha 8, Czech Republic
and

Institute of Thermomechanics, Czech Acad. Sci.
Dolejškova 5, 18200 Praha 8, Czech Republic

GIUSEPPE TOMASSETTI

Università degli Studi Roma Tre, Dipartimento di Ingegneria
Via Vito Volterra 62, 00146 Roma, Italy

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ABSTRACT. We present a model for the dynamics of elastic or poroelastic bodies with monopolar repulsive long-range (electrostatic) interactions at large strains. Our model respects (only) locally the non-self-interpenetration condition but can cope with possible global self-interpenetration, yielding thus a certain justification of most of engineering calculations which ignore these effects in the analysis of elastic structures. These models necessarily combines Lagrangian (material) description with Eulerian (actual) evolving configuration evolving in time. Dynamical problems are studied by adopting the concept of nonlocal nonsimple materials, applying the change of variables formula for Lipschitz-continuous mappings, and relying on a positivity of determinant of deformation gradient thanks to a result by Healey and Krömer.

1. Introduction. In most applications of continuum mechanics, mechanical self interactions between two parts of the same body Ω consist only in contact forces exchanged by these parts along their common boundary. There are, however, situations of physical interest such as for instance electrically-charged, self-gravitating, magnetized, or polarized bodies, where mechanical interactions between parts of the same body are non negligible even if these parts are separated by a positive distance.

In these situations, a peculiarity of the ensuing mathematical model is that the equations that govern the evolution of the body are formulated in Lagrangean form,

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* Corresponding author: Tomáš Roubíček.

(i.e. the independent space variable belongs to a fixed reference configuration) while the equations that determine the long-distance self interactions (through the potential of the electrostatic, gravitation, demagnetising or depolarizing field) are formulated in the Eulerian setting (i.e. the independent space variable belongs to the entire space). Combining the Lagrangian and Eulerian descriptions usually requires injectivity of the deformation related with non-selfpenetration, which can be ensured in static or quasistatic problems but seems extremely difficult in dynamical problems if it would be handled with a mathematical rigor. Overcoming this difficulty, most engineering calculations under large strains ignores non-selfpenetration, too.

To further clarify the issues related to a possible non-invertibility of the deformation, let us consider, as a prototype of body supporting long-range self interactions of monopolar type, for instance an electrically-charged body occupying in its reference configuration a regular region Ω of \mathbb{R}^d , where d is the space dimension. The most natural way to specify the charge content of the body (just like mass content) is through a *referential charge density* $q : \Omega \rightarrow \mathbb{R}$, whose value $q(x)$ at a referential point $x \in \Omega$ is the amount of charge per unit volume at that point when the body is in its undeformed state.

When the body undergoes a deformation $\chi : \Omega \rightarrow \mathbb{R}^d$, the charge bound to the material points of the body undergoes a rearrangement. The energetic cost associated to such rearrangement is the integral over \mathbb{R}^d of the squared gradient of a scalar potential determined by solving the Poisson equation with source term a *spatial charge density* $\mathfrak{q} : \mathbb{R}^d \rightarrow \mathbb{R}$ supported on $\chi(\Omega)$. If the deformation is smooth and invertible both locally and globally, in the sense that it is one-to-one and its Jacobian $J = \det \nabla \chi$ is bounded from below by a positive constant, then the spatial charge density at a given point $\mathbf{x} \in \mathbb{R}^d$ is given by $\mathfrak{q}(\mathbf{x}) = q(\chi^{-1}(\mathbf{x})) / \det(\nabla \chi(\chi^{-1}(\mathbf{x})))$ if $\mathbf{x} \in \chi(\Omega)$, and $\mathfrak{q}(\mathbf{x}) = 0$ otherwise. Since the standard formulation of the boundary-value problem that governs nonlinear elasticity does not ensure injectivity of the deformation map (the model does not incorporate the physics of self contact), we face an issue when willing to give sense to the notion of spatial charge density.

Similar issues are encountered in bodies supporting long-distance self interaction of dipolar type. Although the mechanical treatment of these types of interactions is even more subtle than those of monopolar type, as pointed out in [6], the relevant mathematical literature appears richer. In the static and quasistatic cases, when equilibria are sought through energy minimization, non self-penetration can be ensured by enforcing the Ciarlet-Nečas condition [5]. This fact was exploited, for example, in the paper [23], which contains the analysis of the variational problem that governs equilibrium configurations of polarized and magnetized elastic bodies. Existence results are also offered in [17] in the incompressible case. An existence theorem for magnetized and polarized body can be found in [28], along with a characterization of the conditions under which the stored energy is polyconvex.

In dynamical problems, on the other hand, the device of energy minimization is of no use so that, in particular, the mentioned Ciarlet-Nečas cannot be exploited, and handling non-selfpenetration with mathematical rigor would necessarily require the addition of further ingredients that take into account self contact. For time-independent problems in second-gradient elasticity, an result from [21] is available, showing that there exist weak solutions which satisfy the constraint of injectivity, provided that equilibria are attained with the help of a normal reaction in the form

of a Radon measure supported on the boundary. In dynamics, however, no such result is available to our knowledge.

This issue of non-invertibility of the deformation map was already pointed out in a previous paper of ours [27] and prompted us for further investigations. In the present undertaking we propose a mathematical formulation which, by means of Federer's change-of-variable formula [15], does not rely on the injectivity of the deformation map.

Of course, because of the inherent features of classical elastodynamics whose particular hyperbolic structure prevents a proper mathematical treatment [1], we are anyhow to add further ingredients to conventional nonlinear elasticity. Precisely, we include a regularization which consists in a non-local energy that depends on the second gradient of the deformation map. We refer to the resulting model as a non-local non-simple material. Beside the important analytical regularizing property for the otherwise nonlinear hyperbolic system, one motivation of such nonlocal nonsimple material-concept is dispersion of elastic waves with a large degrees of freedom for covering both normal and anomalous dispersion; cf. the analysis in [16] performed at small strains. Further discussions on nonlocal theories can be found, for example, in [9].

The singularity of the kernel of the nonlocal term is chosen to guarantee that deformations with finite energy belong to a suitable fractional Sobolev space. Such device has two benefits: first, by as a consequence of a result of T. Healey and S. Kroemer [14], the determinant of the deformation gradient is bounded from below by a positive constant; second, since the deformation is Lipschitz continuous, it is possible to apply the aforementioned change-of-variable theorem, which is intimately related to the area formula [15], to provide an alternative formulation of the equations that govern the scalar potential that carries the information about long-distance interactions. This fact constitutes a novelty with respect to our previous treatment in [27], where self interactions were not accounted for in the dynamic setting. The choice of a fractional Sobolev space rather than an integer type permits us to use a quadratic regularization of the stored energy, which is crucial to keep linearity of the term that contains higher-order spatial derivatives in the hyperbolic-type evolution equation. (In principle, one can also think about a local quadratic regularization involving the third-order gradient but this would have produced additional complications in the formulation of the boundary conditions.)

Although the formulation we propose applies to both monopolar and dipolar self interactions, in the present paper we limit our analysis to interactions of monopolar type for technical reasons. To be more specific, the term that accounts for monopolar long-distance interactions appears in the force balance in the form of the *first gradient* of the scalar potential, composed with the deformation; for dipolar interactions, the corresponding term involves the *second gradient*. In the first case, by a suitable manipulation, we can cast the long-distance force term in a form that does not involve the gradient of the scalar potential; in the second case, this is not possible.

A further limitation of our analysis is the assumption that the material be an *ideal dielectric* in the sense of [31]. This assumption amount to neglecting the material part of the electromechanical coupling. From the technical standpoint, this limitation is somehow imposed by the mathematical structure of the problem, which makes it difficult to obtain strong convergence of the electric field. Indeed, if we wanted to include electromechanical coupling, we should add to the free energy

a term that depends on the gradient of the electric field, in a manner similar to what we do with the deformation gradient. On the other hand, the ideal dielectric model, notwithstanding its limitations, finds several applications (see for instance [32] for an application to material instabilities).

A related technical issue also forces us to replace the conventional Poisson equation, with a p -regularization, so as to guarantee that in our constructive approach to existence of solutions the approximated scalar potentials converge in the space of continuous functions. When taking into account the electrostatic scalar potential in the mechanical force balance, the potential must be pulled back into the reference configuration. Such pull back operation involves a composition between the scalar potential and the deformation map, which may be very badly if the scalar potential converge strongly only in a Lebesgue space. In fact, we need a stronger convergence in the space of continuous functions.

We leave it as an open problem the application of our formulation within the standard setting of electrostatics, i.e., without the p -regularization.

The plan of our paper is the following: Section 2 exposes the repulsive monopolar long-range interactions when charges are bound to the body, and discuss the abstract structure of the problem. The rigorous analysis as far as an existence of weak solutions is then performed in Section 3 by employing rather constructive Galerkin approximation. It is then generalized for time-evolving monopoles (electric charge) related to some diffusant (in this case, charge can rearrange themselves in the body). This is done in Section 4 by using the concept of poroelastic solids and Biot-like diffusion driven by the gradient of a chemical potential. Yet, as pointed out already by M.A. Biot at al. [4], this poroelastic-like model “is not restricted to the presence of actual pores. The fluid may be in solution in the solid, or may be adsorbed. Such phenomena are usually associated with the concept of capillarity or osmotic pressure.” In the final Section 5, we end the paper by outlining some modifications or generalizations and the difficulties which accompany them.

2. Elastodynamics with monopolar repulsive interactions. As a general typographical convention, we shall use italicised/slanted fonts to denote mathematical objects that pertain to the reference configuration, and upright fonts for objects that pertain to the current configuration. Consistent with this convention we distinguish the referential domain Ω from its image Ω under the deformation map. We shall stick to a similar convention when dealing with differential operators: for example, we will use either ∇ or ∇ when referring to fields whose domain of definition is the reference domain Ω or the physical space \mathbb{R}^d .

Thorough the whole paper, the reference configuration Ω is assumed to be a bounded Lipschitz domain in \mathbb{R}^d , whose boundary we denote by Γ . We use the standard notation for the Lebesgue L^p -spaces and $W^{k,p}$ for Sobolev spaces whose k -th distributional derivatives are in L^p -spaces. We will also use the abbreviation $H^k = W^{k,2}$. In the vectorial case, we will write $L^p(\Omega; \mathbb{R}^d) \cong L^p(\Omega)^d$ and $W^{1,p}(\Omega; \mathbb{R}^d) \cong W^{1,p}(\Omega)^d$. Also, we admit k noninteger with the reference to the Sobolev-Slobodetskii spaces. Note that, in this notation, we have the compact embedding $H^{2+\gamma}(\Omega) \subset W^{2,p}(\Omega)$ if $p > 2d/(d - 2\gamma)$ and $W^{2,p}(\Omega) \subset W^{1,p^*}(\Omega)$. In particular $H^{2+\gamma}(\Omega) \subset C^1(\bar{\Omega})$ if $d < p < 2d/(d - 2\gamma)$, which can be satisfied if $\gamma > d/2 - 1$ as employed in (29c) to facilitate usage of the results from [14], cf. (40) below. We also denote by meas_d the d -dimensional Hausdorff measure.

On the time interval $I = [0, T]$, we consider the Bochner spaces $L^p(I; X)$ of Bochner measurable mappings $I \rightarrow X$ whose norm is in $L^p(I)$, with X being a Banach space, while $C_w(I; X)$ will denote the Banach space of weakly measurable mappings $I \rightarrow X$. When writing estimates, we denote by C a generic positive constant which may change from one formula to another.

When electrostatic interactions are accounted for, the most general constitutive equation for the free energy includes a dependence on both the deformation and on the electric field (see for example the discussion in Chap.4 of [7], and in particular Eq. 4.38 therein). This general assumption would allow us to incorporate several coupling effects, such as for instance classical piezoelectricity [29, 30]. On the other hand, interesting electromechanical effects can still be captured through the *ideal dielectric model* [31]. In this model, the referential free energy density splits additively in a mechanical part and an electrostatic part. More specifically, this is equivalent to letting the function ϕ in [7, Eq. 4.56] (see also Eq.s 4.37 and 4.38 of the same reference) to depend only on the deformation. The contribution of the electric part of the free energy will be introduced later. As to the mechanical part, we rely on a non-simple material model by defining the mechanical part of the free-energy of the body as

$$\mathcal{E}_{\text{mech}}(\boldsymbol{\chi}) = \int_{\Omega} \varphi(\nabla \boldsymbol{\chi}) \, dx + \mathcal{H}(\nabla^2 \boldsymbol{\chi}), \tag{1}$$

with $\varphi : \text{GL}^+(d) \times \mathbb{R} \rightarrow \mathbb{R}$ a specific free energy with $\text{GL}^+(d) := \{\mathbf{F} \in \mathbb{R}^{d \times d}; \det \mathbf{F} > 0\}$. The quadratic form \mathcal{H} in (1) is defined by

$$\mathcal{H}(\nabla^2 \boldsymbol{\chi}) = \frac{1}{4} \sum_{i=1}^d \int_{\Omega \times \Omega} (\nabla^2 \chi_i(x) - \nabla^2 \chi_i(\tilde{x})) : \mathfrak{K}(x, \tilde{x}) : (\nabla^2 \chi_i(x) - \nabla^2 \chi_i(\tilde{x})) \, d\tilde{x} \, dx \tag{2}$$

with the hyperelastic-moduli symmetric positive-semidefinite kernel $\mathfrak{K} : \Omega \times \Omega \rightarrow \mathbb{R}^{d \times d \times d \times d}$ satisfying $\mathfrak{K}(x, \tilde{x}) = \mathfrak{K}(\tilde{x}, x)$ and with the scalar $\chi_i : \Omega \rightarrow \mathbb{R}$ the i -th component of the deformation map. Thanks to Fubini's theorem, the non-local strain energy has the representation

$$\mathcal{H}(\nabla^2 \boldsymbol{\chi}) = \frac{1}{2} \sum_{i=1}^d \int_{\Omega \times \Omega} \mathfrak{H}_i(\nabla^2 \chi_i) : \nabla^2 \chi_i \, dx, \tag{3}$$

where the second-order tensors $\mathfrak{K}_i, i = 1, \dots, d$ are defined as the Gâteaux differential of \mathcal{H} with respect to the i -th component χ_i of the deformation, namely

$$[\mathfrak{H}_i(\nabla^2 \chi_i)](x) = \int_{\Omega} \mathfrak{K}(x, \tilde{x}) : (\nabla^2 \chi_i(x) - \nabla^2 \chi_i(\tilde{x})) \, d\tilde{x}. \tag{4}$$

This construction automatically ensures frame indifference of the regularizing energy, and if we denote by $\{\mathbf{e}_i\}_{i=1}^d$ the canonical basis of \mathbb{R}^d , then the third-order tensor

$$\mathfrak{H} = \sum_{i=1}^d \mathbf{e}_i \otimes \mathfrak{H}_i \tag{5}$$

is the hyperstress work conjugate of $\nabla^2 \boldsymbol{\chi}$.

As we shall see below with more detail, the kernel \mathfrak{K} of the regularization term $\mathcal{H}(\nabla^2 \boldsymbol{\chi})$ is chosen singular on the diagonal $\{x = \tilde{x}\}$ in such a way to ensure that deformations with bounded energy are in a fractional Sobolev space $H^{2+\gamma}(\Omega; \mathbb{R}^d)$ with $\gamma \geq d/2$. This entails, in particular, that if the deformation $\boldsymbol{\chi}(t)$ at time t

has bounded energy, then $\chi(t) \in W^{2,p}(\Omega; \mathbb{R}^d)$ with $p > d$. Then, as shown in [14], a suitable growth assumption on the bulk free energy φ as the determinant of its deformation gradient tends to zero guarantees that the determinant is uniformly bounded from below away from zero, and hence in particular the deformation is locally invertible.

Let us assume that the body is endowed with a Lagrangian density $q(x)$ of electric charge which is responsible for long-distance interaction, both between the body and the exterior and between the body with itself. If we assume that $q \in L^1(\Omega)$, then q induces a signed measure on Q on the reference configuration in the standard fashion: for every Borel set $\mathcal{P} \subset \Omega$, the quantity $Q(\mathcal{P}) = \int_{\mathcal{P}} q(x) dx$ is the total charge contained in \mathcal{P} .

We stipulate that when the body is set in motion the charges are redistributed in space in a manner described by a set-valued function $\mathbf{Q}(t, P)$ defined in the following fashion: given a *spatial control volume* $P \subset \mathbb{R}^d$, the amount of charge contained in P after the deformation is $\mathbf{Q}(t, P) := Q(\overleftarrow{\chi}(t, P)) = \int_{\overleftarrow{\chi}(t, P)} q(x) dx$, where we use the notation for the set-valued inverse deformation field:

$$\overleftarrow{\chi}(t, P) = \{x \in \Omega : \chi(t, x) \in P\}.$$

In other words, the charge contained in P is the amount of charge continued in the counterimage of P under $\chi(t)$.

Clearly, for this definition to make sense, we must be sure that the counterimage of P is a Lebesgue-measurable set (this is another motivation for putting in the model an ingredient that enforces deformations maps to be regular). This is true, however, because if $\chi \in W^{2,p}(\Omega; \mathbb{R}^d)$ then χ is Lipschitz continuous and hence it maps Borel sets into Borel sets and viceversa. Moreover, it can be shown that $\mathbf{Q}(t, \cdot)$, the Eulerian charge distribution at time t , admits the following density:

$$\mathbf{q}(t, \mathbf{x}) := \sum_{x \in \overleftarrow{\chi}(t, \mathbf{x})} \frac{q(x)}{\det(\nabla \chi(t, x))} \quad \forall (t, \mathbf{x}) \in I \times \mathbb{R}^d. \quad (6)$$

This result is indeed a consequence of a well-known change-of-variables formula due to Federer [15] (see also [10, Thm. 3.9]), which says that for every integrable function f ,

$$\int_{\mathbb{R}^d} \sum_{x \in \overleftarrow{\chi}(t, \mathbf{x})} f(x) dx = \int_{\Omega} f(x) \det(\nabla \chi(t, x)) dx. \quad (7)$$

Let us notice that, in the special case when $f = \mathbf{f} \circ \chi$, with \mathbf{f} a Lebesgue-measurable function on $\Omega(t) = \chi(t, \Omega)$, formula (7) becomes $\int_{\Omega(t)} \mathbf{f} \text{card } \overleftarrow{\chi}(t, \mathbf{x}) dx = \int_{\Omega} \mathbf{f} \circ \chi \det(\nabla \chi(t, x)) dx$. This last formula was proved by Marcus and Mizel in [20] under the sole assumption $\chi(t, \cdot) \in W^{1,p}(\Omega; \mathbb{R}^d)$ (i.e., without requiring Lipschitz continuity of the deformation). Moreover, when (7) is used with $f = 1$, we obtain the Area Formula [15]

$$\int_{\mathbb{R}^d} \text{card} \left(\overleftarrow{\chi}(t, \mathbf{x}) \right) dx = \int_{\Omega} \det(\nabla \chi(t, x)) dx < \infty, \quad (8)$$

a result which implies that the cardinality of the preimage $\overleftarrow{\chi}(t, \mathbf{x})$ is finite for almost all \mathbf{x} , so that the summation in (6) extends over a finite set. In addition to the spatial charge density \mathbf{q} associated to the body charges, we will consider a time-dependent (in our proof later we shall assume that \mathbf{q}_{ext} is time independent)

external charge density $\mathbf{q}_{\text{ext}}(t, \mathbf{x})$. We shall assume that the total density $\mathbf{q} + \mathbf{q}_{\text{ext}}$ determines a potential $\phi(t, x)$ as the solution of the regularized Poisson equation

$$-\operatorname{div}(\varepsilon(|\nabla\phi|)\nabla\phi) = \mathbf{q} + \mathbf{q}_{\text{ext}} \quad \text{in } \mathbb{R}^d, \tag{9}$$

on the whole space, with vanishing conditions at infinity:

$$\lim_{|\mathbf{x}| \rightarrow +\infty} \phi(t, x) = 0. \tag{10}$$

Here we assume that the effective permittivity is

$$\varepsilon(r) = \varepsilon_0(1 + \varepsilon_r|r|^{p-2}) \quad \forall r \geq 0, \tag{11}$$

where ε_0 is the permittivity of vacuum, i.e. $\varepsilon_0 \doteq 8.854 \times 10^{-12} \text{Fm}^{-1}$, and ε_r is a coefficient in a position like the relative (nonlinear) permittivity having the physical dimension $(\text{V/m})^{2-p}$ with $p > d$ is a regularization exponent.

The weak solution of (9)–(10) with \mathbf{q} given by (6) is the unique stationary point with respect to ϕ of the electrostatic energy

$$\mathcal{E}_{\text{elec}}(\boldsymbol{\chi}, \phi) = \int_{\mathbb{R}^d} \left(\sum_{x \in \tilde{\boldsymbol{\chi}}(t, \mathbf{x})} \frac{q(x)}{\det(\nabla\boldsymbol{\chi}(t, x))} \right) \phi(\mathbf{x}) - \frac{\varepsilon_0}{2} |\nabla\phi|^2 - \frac{\varepsilon_1}{p} |\nabla\phi|^p \, d\mathbf{x}, \tag{12}$$

where we have set $\varepsilon_1 = \varepsilon_r \varepsilon_0$.

An application of the change-of-variable formula (7) with $f(x) = q(x)\phi(\boldsymbol{\chi}(t, x))/\det(\nabla\boldsymbol{\chi}(t, x))$ yields (under the assumption that the deformation gradient is Lipschitz-continuous and that determinant of its gradient is bounded from below by a positive constant, the function f is integrable)

$$\begin{aligned} \int_{\mathbb{R}^d} \left(\sum_{x \in \tilde{\boldsymbol{\chi}}(t, \mathbf{x})} \frac{q(x)}{\det(\nabla\boldsymbol{\chi}(t, x))} \right) \phi(t, \mathbf{x}) \, d\mathbf{x} &= \int_{\mathbb{R}^d} \left(\sum_{x \in \tilde{\boldsymbol{\chi}}(x)} \frac{q(x)\phi(t, \boldsymbol{\chi}(t, x))}{\det(\nabla\boldsymbol{\chi}(t, x))} \right) d\mathbf{x} \\ &= \int_{\Omega} q(x)\phi(t, \boldsymbol{\chi}(t, x)) \, dx. \end{aligned} \tag{13}$$

Using the last result and adding the mechanical and electrostatic energy, we obtain the total free energy:

$$\mathcal{E}(\boldsymbol{\chi}, \phi) = \underbrace{\int_{\Omega} \varphi(\nabla\boldsymbol{\chi}) \, d\mathbf{x} + \mathcal{H}(\nabla^2\boldsymbol{\chi})}_{\mathcal{E}_{\text{mech}}(\boldsymbol{\chi})} + \underbrace{\int_{\Omega} q\phi \circ \boldsymbol{\chi} \, d\mathbf{x} - \int_{\mathbb{R}^d} \frac{\varepsilon_0}{2} |\nabla\phi|^2 + \frac{\varepsilon_1}{p} |\nabla\phi|^p \, d\mathbf{x}}_{\mathcal{E}_{\text{elec}}(\boldsymbol{\chi}, \phi)}, \tag{14}$$

To make the comparison with the existing literature easier, we observe that, starting from (14), it is customary to rewrite the electrostatic energy as an integral over the reference configuration. In particular, defining the Lagrangean potential $\phi = \phi \circ \boldsymbol{\chi}$, the total energy would take the form written as

$$\mathcal{E}_{\text{Lagr}}(\boldsymbol{\chi}, \phi) = \int_{\Omega} [\psi(\nabla\boldsymbol{\chi}, \nabla\phi) - q\phi] \, d\mathbf{x} + \mathcal{H}(\nabla^2\boldsymbol{\chi}), \tag{15}$$

with the Lagrangean energy density $\psi(\mathbf{F}, \mathbf{e}) = \varphi(\mathbf{F}) + \varphi_e(\mathbf{e})$ depending separately on the deformation gradient \mathbf{F} and on the Lagrangean electric field $\mathbf{e} = -\nabla\phi$. As pointed out in the Introduction, this assumption corresponds to adopting the ideal dielectric model. Sometimes, \mathcal{E} is called an electrostatic Lagrangean because of its saddle-point-like structure, cf. e.g. [22, Sect.3.2].

We now define the kinetic energy as

$$\mathcal{T}(\dot{\chi}) = \int_{\Omega} \frac{\varrho}{2} |\dot{\chi}|^2 dx, \tag{16}$$

where $\varrho = \varrho(x)$ with $\inf_{x \in \Omega} \varrho(x) > 0$ is the mass density. The (conventional) Lagrangean is then defined as

$$\mathcal{L}(\chi, \dot{\chi}, \phi) := \mathcal{T}(\dot{\chi}) - \mathcal{E}(\chi, \phi). \tag{17}$$

Considering still an external electromechanical loading $\mathcal{F} = (\mathcal{F}_m(t), \mathcal{F}_e(t))$ defined as a linear functional

$$\langle \mathcal{F}_m(t), \chi \rangle = \int_{\Omega} \mathbf{f}(t) \cdot \chi dx + \int_{\Gamma} \mathbf{g}(t) \cdot \chi dS \quad \text{and} \quad \langle \mathcal{F}_e(t), \phi \rangle = \int_{\mathbb{R}^d} \mathbf{q}_{\text{ext}} \phi dx, \tag{18}$$

the governing equation(s) can then be derived by the *Hamilton variational principle* which asserts that, among all admissible motions on a fixed time interval $[0, T]$, the actual motion is such that the integral

$$\int_0^T \mathcal{L}(\chi(t), \dot{\chi}(t), \phi(t)) + \langle \mathcal{F}(t), (\chi, \phi)(t) \rangle dt \quad \text{is stationary,} \tag{19}$$

i.e. (χ, ϕ) is its *critical point*. This gives the system

$$\mathcal{T}' \ddot{\chi} + \partial_{\chi} \mathcal{E}(\chi, \phi) = \mathcal{F}_m, \tag{20a}$$

$$\partial_{\phi} \mathcal{E}(\chi, \phi) = \mathcal{F}_e, \tag{20b}$$

where \mathcal{T}' denotes the Gâteaux derivative of the quadratic functional \mathcal{T} , i.e. a linear operator, and $\partial_{\chi} \mathcal{E}$ and $\partial_{\phi} \mathcal{E}$ are the respective partial Gâteaux differentials. The abstract “algebraic” part (20b) forms a *holonomic constraint*.

Taking into account the specific forms of the energies (14) and (16), the abstract system (20) results to

$$\varrho \ddot{\chi} - \operatorname{div} \mathbf{S} + q \nabla \phi \circ \chi = \mathbf{f} \quad \text{with} \quad \mathbf{S} = \varphi'(\nabla \chi) - \operatorname{div} \mathfrak{H}(\nabla^2 \chi), \tag{21a}$$

$$\operatorname{div}((\varepsilon_0 + \varepsilon_1 |\nabla \phi|^{p-2}) \nabla \phi) + \sum_{x \in \bar{\chi}(\cdot, t)} \frac{q(x)}{\det(\nabla \chi(t, x))} + \mathbf{q}_{\text{ext}}(t, \cdot) = 0 \tag{21b}$$

for a.a. $t \in I$, with $\nabla \phi$ understood as space derivatives in the actual space configuration while other time/space derivatives are in the reference configuration, and the nonlocal hyperstress $\mathfrak{H}(\nabla^2 \chi)$ is given by (4) and (5). Note that the force in (21a) can be seen from (14) while the right-hand side of (21b) can be seen from the equivalent form (1). This system is augmented with the following boundary conditions

$$\mathbf{S} \mathbf{n} - \operatorname{div}_s \mathfrak{H}(\nabla^2 \chi) = \mathbf{g} \quad \text{and} \quad \mathfrak{H}(\nabla^2 \chi) : (\mathbf{n} \otimes \mathbf{n}) = 0 \quad \text{on} \quad \Sigma \tag{22a}$$

$$\lim_{|x| \rightarrow \infty} \phi(t, \mathbf{x}) = 0 \quad \text{for all } t \in (0, T). \tag{22b}$$

and by the initial conditions

$$\chi(\cdot, 0) = \chi_0, \quad \dot{\chi}(\cdot, 0) = \mathbf{v}_0 \quad \text{in} \quad \Omega. \tag{23}$$

As there is no dissipation energy, the system (20) is (at least formally) conservative, i.e. it conserves the mechanical energy $\mathcal{T}(\dot{\chi}) + \mathcal{E}(\chi, \phi)$ during the evolution provided that the forcing term vanishes: $\mathcal{F} = 0$. As there no kinetic energy associated with ϕ -variable, (20) has the structure of an *abstract differential-algebraic equation* (DAE) with ϕ the “fast” variable and χ the “slow” variable. We conclude

this section by showing that this system can be reduced to a single evolutionary partial differential equation for the deformation χ .

To begin with, we observe that $\mathcal{E}(\chi, \cdot)$ is strictly concave. This feature makes it possible for us to solve (20b) with respect to ϕ . The most convenient way to achieve this goal by means of the Legendre transform More specifically, we introduce the quantity

$$\mathcal{E}_{\text{tot}}(\chi, \phi) = \mathcal{E}(\chi, \phi) - \langle \mathcal{F}, (\chi, \phi) \rangle, \tag{24}$$

and then we let

$$[-\mathcal{E}_{\text{tot}}(\chi, \cdot)]^*(\chi, \xi) = \sup_{\phi \in \mathbb{R}} (\langle \xi, \phi \rangle + \mathcal{E}_{\text{tot}}(\chi, \phi)), \tag{25}$$

where, in the second formula, $\langle \cdot, \cdot \rangle$ denotes the duality between $W^{1,p}(\mathbb{R}^d)$ and $W^{-1,p}(\mathbb{R}^d)$. Given $\xi \in W^{-1,p}(\mathbb{R}^d)$, let us define ϕ_ξ to be the solution of the following problem:

$$\varepsilon_0 \Delta \phi_\xi + \varepsilon_1 \operatorname{div}(|\nabla \phi_\xi|^{p-2} \nabla \phi_\xi) + \sum_{x \in \overleftarrow{\chi}(\cdot, t)} \frac{q(x)}{\det(\nabla \chi(t, x))} + \mathbf{q}_{\text{ext}} = -\xi, \tag{26}$$

with boundary conditions vanishing at infinity. Then we have

$$[-\mathcal{E}_{\text{tot}}(\chi, \cdot)]^*(\chi, \xi) = \int_{\mathbb{R}^d} \frac{\varepsilon_0}{2} |\nabla \phi_\xi|^2 + \frac{\varepsilon_1}{p'} |\nabla \phi_\xi|^p dx, \tag{27}$$

a representation formula which shows the convex character of the dual of $\mathcal{E}_{\text{tot}}(\chi, \cdot)$. It is a now standard result from Convex Analysis that

$$\phi = -\partial_\xi [-\mathcal{E}_{\text{tot}}(\chi, 0)]^*,$$

so that, by substitution into (20a)

$$\mathcal{I}' \ddot{\chi} + \partial_\chi \mathcal{E}(\chi, -\partial_\xi [-\mathcal{E}_{\text{tot}}(\chi, 0)]^*) = \mathcal{F}_m. \tag{28}$$

3. Analysis of the model by the Galerkin approximation. An important attribute of the model is that $\det(\nabla \chi)$ occurs in the denominators in (21b). This requires to have control over $1/\det(\nabla \chi)$, which can be ensured by having the free energy $\varphi = \varphi(\mathbf{F})$ blowing up to $+\infty$ sufficiently fast if $\det \mathbf{F} \rightarrow 0+$. More specifically, together with frame indifference, we assume altogether:

$$\varphi : \mathbb{R}^{d \times d} \rightarrow [0, +\infty] \text{ continuously differentiable on } \text{SL}^+(d), \tag{29a}$$

$$\forall \mathbf{Q} \in \text{SO}(d) : \varphi(\mathbf{Q}\mathbf{F}) = \varphi(\mathbf{F}), \tag{29b}$$

$$\varphi(\mathbf{F}) \begin{cases} \geq \epsilon / (\det \mathbf{F})^p & \text{if } \det \mathbf{F} > 0, \\ = +\infty & \text{if } \det \mathbf{F} \leq 0, \end{cases} \text{ for some } p > \frac{2d}{d-2-2\gamma}, \gamma > \frac{d}{2} - 1. \tag{29c}$$

where γ refers to (30). Concerning the regularizing kernel \mathfrak{K} , we assume

$$\exists \epsilon > 0 \forall x, \tilde{x} \in \Omega, G \in \mathbb{R}^{d \times d} : \left(\frac{\epsilon |G|^2}{|x - \tilde{x}|^{d+2\gamma}} - \frac{1}{\epsilon} \right)^+ \leq G : \mathfrak{K}(x, \tilde{x}) : G \leq \frac{|G|^2}{\epsilon |x - \tilde{x}|^{d+2\gamma}}. \tag{30}$$

Furthermore, we shall make the following assumptions concerning the initial data and the body loading and the surface loading

$$\chi_0 \in H^{2+\gamma}(\Omega; \mathbb{R}^d) \text{ with } \varphi(\nabla \chi_0) \in L^1(\Omega), \quad \mathbf{v}_0 \in L^2(\Omega; \mathbb{R}^d), \tag{31a}$$

$$\mathbf{f} \in L^\infty(0, T; L^2(\Omega; \mathbb{R}^d)), \quad \mathbf{g} \in W^{1,1}(0, T; L^1(\Gamma; \mathbb{R}^d)). \tag{31b}$$

The derivation of the weak formulation of (21a) is standard, but requires some care. We take the scalar product of both sides with a test velocity ζ such that $\zeta(T) = \mathbf{0}$ and $\dot{\zeta}(T) = \mathbf{0}$, and we integrate over the domain Ω and on the time interval I . Then we integrate by parts twice both on the domain Ω and on the time interval I . The result is

$$\begin{aligned} & \int_Q \varrho \chi \cdot \ddot{\zeta} + \varphi'(\nabla \chi) : \nabla \zeta + \mathfrak{H}(\nabla^2 \chi) : \nabla^2 \zeta \, dx dt \\ &= - \int_Q q(\nabla \phi \circ \chi) \cdot \zeta \, dx dt + \int_{\Sigma} \mathbf{g} \cdot \zeta \, dS dt + \int_{\Omega} \varrho \mathbf{v}_0 \cdot \zeta(0) - \varrho \chi_0 \cdot \dot{\zeta}(0) \, dx. \end{aligned} \quad (32)$$

In our proof of existence of weak solutions, a passage to the limit in the integral term involving $\nabla \phi$ on the right-hand side would be problematic. Indeed, the term $\nabla \phi$ is composed with the deformation χ , and in order to pass to the limit in our approximation procedure even strong convergence of $\nabla \phi$ in a L^p space would not suffice. Indeed, we would need convergence of $\nabla \phi$ in the space of continuous functions, which however cannot be expected. For this reason, we have to rewrite the aforementioned term through a few manipulations. As a start, we observe that $(\nabla \phi) \circ \chi = \nabla \chi^{-\top} \nabla(\phi \circ \chi)$. Thus, using integration by parts we obtain

$$\begin{aligned} & \int_Q q(\nabla \phi \circ \chi) \cdot \zeta \, dx dt = \int_Q q \zeta \cdot \nabla \chi^{-\top} \nabla(\phi \circ \chi) = \int_Q \nabla(\phi \circ \chi) \cdot \nabla \chi^{-1}(q \zeta) = \\ &= \int_{\Sigma} q(\phi \circ \chi) \nabla \chi^{-\top} \mathbf{n} \cdot \zeta - \int_Q \phi \circ \chi \operatorname{div}(q \nabla \chi^{-1} \zeta) \, dx dt \\ &= \int_{\Sigma} q(\phi \circ \chi) \nabla \chi^{-\top} \mathbf{n} \cdot \zeta - \int_Q (\phi \circ \chi) \nabla \chi^{-\top} \cdot \nabla(q \zeta) \, dx dt \\ & \quad - \int_Q q(\phi \circ \chi) \zeta \cdot \operatorname{div}(\nabla \chi^{-\top}) \, dx dt. \end{aligned} \quad (33)$$

We can now write our notion of weak solution.

Definition 3.1 (Weak solution to the initial-boundary-value problem (21)–(22)–(23)). A pair $(\chi, \phi) \in C_w(I; H^{2+\gamma}(\Omega; \mathbb{R}^d)) \times L^\infty(I; W^{1,p}(\mathbb{R}^d))$ is a weak solution to (21), (22), and (23) if $\mathcal{H}(\nabla^2 \chi) \in L^\infty(I)$, if $\dot{\chi} \in C_w(I; L^2(\Omega; \mathbb{R}^d))$, and if the following conditions hold:

1) for every $\zeta \in C^\infty(\overline{Q}; \mathbb{R}^d)$ satisfying $\zeta(T) = \dot{\zeta}(T) = 0$,

$$\begin{aligned} & \int_Q \varrho \chi \cdot \ddot{\zeta} + \varphi'(\nabla \chi) : \nabla \zeta + \mathfrak{H}(\nabla^2 \chi) : \nabla^2 \zeta(x) \, dx dt = \int_Q (\phi \circ \chi) \nabla \chi^{-\top} : \nabla(q \zeta) \\ & + \left(q(\phi \circ \chi) \operatorname{div}(\nabla \chi^{-\top}) + \mathbf{f} \right) \cdot \zeta \, dx dt + \int_{\Sigma} (q(\phi \circ \chi) \nabla \chi^{-\top} \mathbf{n} + \mathbf{g}) \cdot \zeta \, dS dt \\ & \quad + \int_{\Omega} \varrho \mathbf{v}_0 \cdot \zeta(0) - \varrho \chi_0 \cdot \dot{\zeta}(0) \, dx, \end{aligned} \quad (34a)$$

with the hyperstress

$$[\mathfrak{H}(\nabla^2 \chi)](t, x) = \sum_{i=1}^d \mathbf{e}_i \otimes \int_{\Omega} \mathfrak{K}_i(x, x') : (\nabla^2 \chi_i(t, x') - \nabla^2 \chi_i(t, x)) \, dx'$$

a.e. on Q , cf. (4) and (5).

2) For every $\zeta \in C_0^\infty(I \times \mathbb{R}^d)$,

$$\int_{I \times \mathbb{R}^d} (\varepsilon_0 + \varepsilon_1 |\nabla \phi|^{p-2}) \nabla \phi \cdot \nabla \zeta - \mathbf{q}_{\text{ext}} \zeta \, dx dt = \int_Q q(\zeta \circ \chi) \, dx dt. \tag{34b}$$

If φ were semiconvex, that is, if φ'' was bounded from below, we could use a standard technique (see for instance [18, Remark 9.5.4]), based on the a time discretization to provide a constructive proof of existence of solutions. However, in the present case semiconvexity is incompatible with the frame-indifference requirement (29b). Instead, we will resort to the Galerkin method to construct approximate solutions. It is important that the singularity of the free energy for $\det \mathbf{F} \rightarrow 0+$ is eliminated together with the spurious Lavrentiev phenomenon (cf. [11]) by the used nonsimple-material concept in cooperation with the Healey-Krömer theorem [14]. It is important that this can be done already on the level of Galerkin approximation, so that no other regularization of the singularity of φ at $\det \mathbf{F} \rightarrow 0+$ is not needed, cf. also [27].

To this goal, we take nested (with respect to “ \subset ”) sequences (indexed by $k \in \mathbb{N}$) of some finite-dimensional subspaces X_k of $H^{2+\gamma}(\Omega; \mathbb{R}^d)$ whose union is dense, i.e.:

$$X_k \subset X_{k+1}, \quad \overline{\bigcup_k X_k} = H^{2+\gamma}(\Omega; \mathbb{R}^d). \tag{35}$$

Without loss of generality, we may assume that X_0 is spanned by the initial configuration $\chi_0 \in H^{2+\gamma}(\Omega; \mathbb{R}^d)$ and by the initial velocity $\mathbf{v}_0 \in L^2(\Omega; \mathbb{R}^d)$. Then for each $k \in \mathbb{N}$ we solve the following approximation of (34), which we write in an abstract form, in the same fashion of (20):

$$\langle \mathcal{F}' \ddot{\chi}_k + \partial_{\chi} \mathcal{E}(\chi_k, \phi_k), \mathbf{v} \rangle = \langle \mathcal{F}_m, \mathbf{v} \rangle \quad \forall \mathbf{v} \in X_k, \tag{36a}$$

$$\partial_{\phi} \mathcal{E}(\chi_k, \phi_k) - \mathcal{F}_e = 0. \tag{36b}$$

It should be pointed out that the Galerkin approximation applies only to (36a) while (36b) is kept continuous. The existence of solutions for the above system can be obtained by following a path similar to that leading to (28), which leads to

$$\left\langle \mathcal{F}' \ddot{\chi}_k + \partial_{\chi} \mathcal{E} \left(\chi_k, -\partial_{\xi} [-\mathcal{E}_{\text{tot}}(\chi_k, 0)]^* \right), \mathbf{v} \right\rangle = 0 \quad \forall \mathbf{v} \in X_k, \tag{37}$$

which is equivalent to a system of ordinary differential equations whose solution for small times can be proved by standard arguments. When the solution of (37) is obtained, we define the approximants ϕ_k as

$$\phi_k = -\partial_{\xi} [-\mathcal{E}_{\text{tot}}(\chi_k, 0)]. \tag{38}$$

The following result by T.J. Healey and S. Krömer [14] is of an essential importance: Formulating it in an arbitrary dimension $d \in \mathbb{N}$, let $p > d$, $q \geq pd/(p-d)$, and $K \in \mathbb{R}$, then there is $\varepsilon = \varepsilon(p, q, K) > 0$ such that

$$\left. \begin{aligned} &\chi \in W^{2,p}(\Omega; \mathbb{R}^d) \\ &\det(\nabla \chi) > 0 \text{ a.e. in } \Omega \\ &\|\chi\|_{W^{2,p}(\Omega; \mathbb{R}^d)} + \int_{\Omega} 1/(\det \nabla \chi(x))^q \, dx \leq K \end{aligned} \right\} \Rightarrow \det(\nabla \chi) \geq \varepsilon \text{ on } \bar{\Omega}. \tag{39}$$

Here we will use it in a modification involving the quadratic form \mathcal{H} as in (2) with (30) satisfied for some $\gamma > d/2 - 1$ and with a potential $\varphi : \mathbb{R}^{d \times d} \rightarrow \mathbb{R}$ satisfying $\varphi(\mathbf{F}) \geq 1/(\det \mathbf{F})^q$ for $q > 2d/(2\gamma+2-d)$, cf. [18, Sect. 2.5]: Considering

the functional $\Phi(\boldsymbol{\chi}) := \int_{\Omega} \varphi(\nabla \boldsymbol{\chi}) \, dx + \mathcal{H}(\nabla^2 \boldsymbol{\chi})$, for any K , there is $\varepsilon > 0$ such that

$$\Phi(\boldsymbol{\chi}) \leq K \quad \Rightarrow \quad \det(\nabla \boldsymbol{\chi}) \geq \varepsilon \quad \text{on } \bar{\Omega}. \quad (40)$$

Proposition 1 (Weak solutions to (21)). *Assume that $q \in W^{1,1}(\Omega)$, that $\mathbf{q}_{\text{ext}} \in L^1(\mathbb{R}^d)$ is time independent, and that $p > d$. Then the Galerkin approximation $(\boldsymbol{\chi}_k, \phi_k)$ exists on the whole time interval I and satisfies the a-priori estimates*

$$\|\boldsymbol{\chi}_k\|_{W^{1,\infty}(I;L^2(\Omega;\mathbb{R}^d)) \cap L^\infty(I;H^{2+\gamma}(\Omega;\mathbb{R}^d))} \leq C \quad \text{with} \quad \left\| \frac{1}{\det(\nabla \boldsymbol{\chi}_k)} \right\|_{L^\infty(Q)} \leq C, \quad (41a)$$

$$\|\phi_k\|_{L^\infty(I;W^{1,p}(\mathbb{R}^d))} \leq C. \quad (41b)$$

Moreover, there is a subsequence of $\{(\boldsymbol{\chi}_k, \phi_k)\}_{k \in \mathbb{N}}$ converging weakly* in the topologies indicated in (41) and the limit of any such subsequence is the weak solution to (21) with the initial conditions $\boldsymbol{\chi}(0) = \boldsymbol{\chi}_0$ and $\dot{\boldsymbol{\chi}} = \mathbf{v}_0$.

Proof. By the Healey-Krömer theorem [14] in its modification as (40), for some positive $\varepsilon \leq \min_{x \in \bar{\Omega}} \det \nabla \boldsymbol{\chi}_0(x)$, we have

$$\forall (t, x) \in \bar{Q} : \quad \det \nabla \boldsymbol{\chi}_k(t, x) \geq \varepsilon. \quad (42)$$

For (42), we use the successive-continuation argument on the Galerkin level, and thus $\nabla \boldsymbol{\chi}_k$ is valued in the definition domain φ and the singularity of φ is not seen. In particular, the Lavrentiev phenomenon (which may occur when φ would not have enough fast growth to $+\infty$ if $\det \mathbf{F} \rightarrow 0+$) is excluded.

Having a local solution of the Galerkin approximation, we test (36a) and (36b) by $\dot{\boldsymbol{\chi}}_k$ and $\dot{\phi}_k$, respectively, we add the resulting equations and we use the chain rule to obtain

$$\begin{aligned} & \int_{\Omega} \frac{\rho}{2} |\dot{\boldsymbol{\chi}}_k(\tau)|^2 \, dx + \mathcal{E}_{\text{mech}}(\boldsymbol{\chi}_k(\tau)) + \mathcal{E}_{\text{elec}}(\boldsymbol{\chi}_k(\tau), \phi_k(\tau)) = \int_{\Omega} \frac{\rho}{2} |\dot{\mathbf{v}}_0|^2 \, dx \\ & + \mathcal{E}(\boldsymbol{\chi}_0, \phi_0) + \int_0^\tau \int_{\Omega} \mathbf{f} \cdot \dot{\boldsymbol{\chi}}_k \, dx \, dt + \int_0^\tau \int_{\Gamma} \mathbf{g} \cdot \dot{\boldsymbol{\chi}}_k \, dS \, dt + \int_0^\tau \int_{\mathbb{R}^d} \mathbf{q}_{\text{ext}} \dot{\phi}_k \, dx \, dt \end{aligned}$$

for all τ in some interval $(0, T_k)$ with $T_k \leq T$. Then, we integrate by parts with respect to time the terms containing time derivatives of $\boldsymbol{\chi}_k$ and ϕ_k that cannot be controlled by the energetic terms, namely, the terms involving \mathbf{g} and \mathbf{q}_{ext} , and we obtain

$$\begin{aligned} & \int_{\Omega} \frac{\rho}{2} |\dot{\boldsymbol{\chi}}_k(\tau)|^2 \, dx + \mathcal{E}_{\text{mech}}(\boldsymbol{\chi}_k(\tau)) - \int_{\Sigma} \mathbf{g}(\tau) \cdot \boldsymbol{\chi}_k(\tau) \, dS + \mathcal{E}_{\text{elec}}(\boldsymbol{\chi}_k(\tau), \phi_k(\tau)) \\ & - \int_{\mathbb{R}^d} \mathbf{q}_{\text{ext}}(\tau) \phi_k(\tau) \, dx = \int_{\Omega} \frac{\rho}{2} |\dot{\mathbf{v}}_0|^2 \, dx + \mathcal{E}(\boldsymbol{\chi}_0, \phi_0) + \int_0^\tau \int_{\Omega} \mathbf{f} \cdot \dot{\boldsymbol{\chi}}_k \, dx \, dt \\ & - \int_0^\tau \int_{\Gamma} \dot{\mathbf{g}} \cdot \boldsymbol{\chi}_k \, dS \, dt - \int_{\Sigma} \mathbf{g}(0) \cdot \boldsymbol{\chi}_k(0) \, dS - \int_{\mathbb{R}^d} \mathbf{q}_{\text{ext}}(0) \phi_k(0) \, dx. \quad (43) \end{aligned}$$

We observe that, for every $\delta > 0$,

$$\begin{aligned} & \mathcal{E}_{\text{mech}}(\boldsymbol{\chi}_k(\tau)) - \int_{\Gamma} \mathbf{g}(\tau) \cdot \boldsymbol{\chi}_k(\tau) \\ & \geq C_1 \|\nabla^2 \boldsymbol{\chi}_k(\tau)\|_{L^2(\Omega; \mathbb{R}^{d \times d \times d})} - \frac{1}{2\delta} \|\mathbf{g}(\tau)\|_{L^1(\Gamma; \mathbb{R}^d)}^2 - \frac{\delta}{2} \|\boldsymbol{\chi}_k(\tau)\|_{L^\infty(\Omega; \mathbb{R}^d)}^2 \\ & \geq C_1 \|\boldsymbol{\chi}_k(\tau)\|_{H^{2+\gamma}(\Omega; \mathbb{R}^d)}^2 - C_2 \|\boldsymbol{\chi}_k(\tau)\|_{L^2(\Omega; \mathbb{R}^d)}^2 \\ & \quad - \frac{1}{2\delta} \|\mathbf{g}(\tau)\|_{L^1(\Gamma; \mathbb{R}^d)}^2 - \frac{\delta}{2} \|\boldsymbol{\chi}_k(\tau)\|_{L^\infty(\Omega; \mathbb{R}^d)}^2. \end{aligned}$$

From the above equation, by using the embedding $H^{2+\gamma}(\Omega) \subset C(\bar{\Omega})$, and by taking δ sufficiently small, we obtain

$$\mathcal{E}_{\text{mech}}(\boldsymbol{\chi}_k(\tau)) - \int_{\Gamma} \mathbf{g}(\tau) \cdot \boldsymbol{\chi}_k(\tau) \, dS \geq C_1 \|\boldsymbol{\chi}_k\|_{H^{2+\gamma}(\Omega; \mathbb{R}^d)}^2 - C_2 \left(\|\boldsymbol{\chi}_k\|_{L^2(\Omega; \mathbb{R}^d)}^2 + 1 \right). \tag{44}$$

Next, we notice that if $\phi_k(t)$ solves the nonlinear electrostatic equation (36b) at time t , then a test by $\phi_k(t)$ yields

$$\int_{\Omega} q \phi_k(t) \circ \boldsymbol{\chi}_k(t) \, dx + \int_{\mathbb{R}^d} \mathbf{q}_{\text{ext}}(t) \phi_k(t) \, dx = \int_{\mathbb{R}^d} \varepsilon_0 |\nabla \phi_k(t)|^2 + \varepsilon_1 |\nabla \phi_k(t)|^p \, dx. \tag{45}$$

This implies that

$$\mathcal{E}_{\text{elec}}(\boldsymbol{\chi}_k(t), \phi_k(t)) - \int_{\mathbb{R}^d} \mathbf{q}_{\text{ext}}(\tau) \phi_k(\tau) \, dx = \int_{\mathbb{R}^d} \frac{\varepsilon_0}{2} |\nabla \phi_k(t)|^2 + \frac{\varepsilon_1}{p'} |\nabla \phi_k(t)|^p \, dx. \tag{46}$$

Furthermore, we have

$$\begin{aligned} & \int_0^\tau \int_{\Omega} \mathbf{f} \cdot \dot{\boldsymbol{\chi}}_k \, dx \, dt \leq \|\mathbf{f}\|_{L^\infty(I; L^2(\Omega; \mathbb{R}^d))} \int_0^\tau \|\dot{\boldsymbol{\chi}}_k\|_{L^2(\Omega; \mathbb{R}^d)} \, dt \\ & \leq \|\mathbf{f}\|_{L^\infty(I; L^2(\Omega; \mathbb{R}^d))} + \frac{\tau}{2} + \frac{1}{2} \int_0^\tau \|\dot{\boldsymbol{\chi}}_k\|_{L^2(\Omega; \mathbb{R}^d)}^2 \, dt \leq C \left(1 + \int_0^\tau \|\dot{\boldsymbol{\chi}}_k\|_{L^2(\Omega; \mathbb{R}^d)}^2 \, dt \right). \end{aligned} \tag{47}$$

Likewise, we have

$$\begin{aligned} & - \int_0^\tau \int_{\Gamma} \dot{\mathbf{g}} \cdot \boldsymbol{\chi}_k \, dS \, dt \leq \int_0^\tau \|\dot{\mathbf{g}}\|_{L^1(\Gamma; \mathbb{R}^d)} \|\boldsymbol{\chi}_k\|_{L^\infty(\Gamma; \mathbb{R}^d)} \, dt \\ & \leq \frac{1}{2} \int_0^\tau \|\dot{\mathbf{g}}\|_{L^1(\Gamma; \mathbb{R}^d)}^2 \, dt + \frac{1}{2} \int_0^\tau \|\boldsymbol{\chi}_k\|_{L^\infty(\Gamma; \mathbb{R}^d)}^2 \, dt \leq C \left(1 + \int_0^\tau \|\boldsymbol{\chi}_k\|_{H^{2+\gamma}(\Omega; \mathbb{R}^d)}^2 \, dt \right). \end{aligned} \tag{48}$$

By combining the estimates (46), (47), and (48) we obtain

$$\begin{aligned} & \|\dot{\boldsymbol{\chi}}_k(\tau)\|_{L^2(\Omega; \mathbb{R}^d)}^2 + \|\boldsymbol{\chi}_k(\tau)\|_{H^{2+\gamma}(\Omega; \mathbb{R}^d)}^2 + \|\Phi_k(\tau)\|_{W^{1,p}(\mathbb{R}^d)}^p \\ & \leq C \left(1 + \|\boldsymbol{\chi}_k(\tau)\|_{L^2(\Omega; \mathbb{R}^d)}^2 + \int_0^\tau \|\dot{\boldsymbol{\chi}}_k(t)\|_{L^2(\Omega; \mathbb{R}^d)}^2 \, dt \right). \end{aligned}$$

Finally, we observe that

$$\begin{aligned} \|\boldsymbol{\chi}_k(\tau)\|_{L^2(\Omega; \mathbb{R}^d)}^2 & = \int_{\Omega} \left| \boldsymbol{\chi}_0 + \int_0^\tau \dot{\boldsymbol{\chi}}_k(t) \, dt \right|^2 \, dx \\ & \leq 2 \int_{\Omega} |\boldsymbol{\chi}_0|^2 \, dx + 2 \int_{\Omega} \left| \int_0^\tau \dot{\boldsymbol{\chi}}_k(t) \, dt \right|^2 \, dx \\ & \leq 2 \int_{\Omega} |\boldsymbol{\chi}_0|^2 \, dx + 2\tau \int_{\Omega} \int_0^\tau |\dot{\boldsymbol{\chi}}_k(t)|^2 \, dt \, dx. \end{aligned} \tag{49}$$

Thus, we can estimate

$$\begin{aligned} & \|\dot{\chi}_k(\tau)\|_{L^2(\Omega;\mathbb{R}^d)}^2 + \|\chi_k(\tau)\|_{H^{2+\gamma}(\Omega;\mathbb{R}^d)}^2 + \|\phi_k(\tau)\|_{W^{1,p}(\mathbb{R}^d)}^p \\ & \leq C \left(1 + \int_0^\tau \|\dot{\chi}_k(t)\|_{L^2(\Omega;\mathbb{R}^d)} + \|\chi_k(t)\|_{H^{2+\gamma}(\Omega;\mathbb{R}^d)}^2 \right). \end{aligned} \tag{50}$$

At this point, by the application of Gronwall’s inequality and by the standard continuation argument we can deduce the existence of approximate solutions on the whole time interval I , along with the bounds (41a) and (41b).

By Banach’s selection principle, there exist $\chi \in L^\infty(I; H^{2+\gamma}(\Omega; \mathbb{R}^d)) \cap W^{1,\infty}(I; L^2(\Omega; \mathbb{R}^d))$ and $\phi \in L^\infty(I; W^{1,p}(\mathbb{R}^d))$ such that, by possibly extracting a subsequence (which we do not relabel),

$$\chi_k \rightarrow \chi \quad \text{weakly* in } L^\infty(I; H^{2+\gamma}(\Omega; \mathbb{R}^d)) \cap W^{1,\infty}(I; L^2(\Omega; \mathbb{R}^d)), \tag{51a}$$

$$\phi_k \rightarrow \phi \quad \text{weakly* in } L^\infty(I; W^{1,p}(\Omega)). \tag{51b}$$

By the compact embedding $H^{2+\gamma}(\Omega) \subseteq C^1(\overline{\Omega})$ and by the continuous embedding $C^1(\overline{\Omega}) \subset L^2(\overline{\Omega})$, the Aubin-Lions theorem and (51a) imply that

$$\chi_k \rightarrow \chi \quad \text{strongly in } L^r(I; C^1(\overline{\Omega}; \mathbb{R}^d)) \quad \forall 1 \leq r < \infty. \tag{52}$$

This also implies that the restriction of $\nabla \chi_k$ on Γ converges strongly in $L^r(I; C(\Gamma; \mathbb{R}^{d \times d}))$. Consider the function $\mathbf{F} \mapsto \mathbf{F}^{-\top} = \text{Cof } \mathbf{F} / \det \mathbf{F}$ on the set of all $d \times d$ -matrices whose determinant is bounded from below by a positive constant. This function induces a continuous map from $C(\overline{\Omega}; \mathbb{R}^{d \times d})$ to itself. Hence, from (52) we have

$$\nabla \chi_k^{-\top} \rightarrow \nabla \chi^{-\top} \quad \text{strongly in } L^r(I; C(\overline{\Omega}; \mathbb{R}^{d \times d})) \quad \forall 1 \leq r < \infty. \tag{53}$$

Next, it follows from the $W^{1,\infty}$ -estimate in (41a) that

$$\|\nabla^2 \dot{\chi}_k\|_{L^\infty(I; H^2(\Omega; \mathbb{R}^{d \times d \times d}))} \leq C. \tag{54}$$

Also, we have that $H^\gamma(\Omega)$ is compactly embedded in $L^2(\Omega)$. Thus, in view of the L^∞ estimate in (41a) we can again apply the Aubin-Lions theorem to $\nabla^2 \chi$ to deduce that

$$\nabla^2 \chi_k \rightarrow \nabla^2 \chi \quad \text{strongly in } L^r(I; L^2(\Omega; \mathbb{R}^{d \times d \times d})) \quad \forall 1 \leq r < \infty. \tag{55}$$

The strong convergence statement (52) can be written as $\int_0^T \|\chi_k(t) - \chi(t)\|_{C^1(\overline{\Omega}; \mathbb{R}^d)}^r \rightarrow 0$ as $k \rightarrow \infty$. This also implies that

$$\chi_k(t) \rightarrow \chi(t) \quad \text{strongly in } C^1(\overline{\Omega}; \mathbb{R}^d) \text{ for a.a. } t \in (0, T). \tag{56}$$

We also know from the uniform estimate (41b) that for each k there exists a set I_k such that $\text{meas}(I \setminus I_k) = 0$ and such that $\|\phi_k(t)\|_{W^{1,p}(\mathbb{R}^d)} \leq C$ for all $t \in I_k$. We let $J = \cap_k I_k$. Then $\text{meas}(I \setminus J) = 0$, and $\|\phi_k(t)\|_{W^{1,p}(\mathbb{R}^d)} \leq C$ for all $t \in J$, for all indices k . Now, by passing to a subsequence, we have that for every closed ball $B \subset \mathbb{R}^d$

$$\phi_k(t) \rightarrow \phi(t) \quad \text{strongly in } C(B) \text{ for a.a. } t \in (0, T). \tag{57}$$

The functions ϕ_k are solution of the weak equation

$$\int_{\mathbb{R}^d} (\varepsilon_0 + \varepsilon_1 |\nabla \phi_k(t)|^{p-2}) \nabla \phi_k(t) \cdot \nabla \zeta \, dx = \int_{\mathbb{R}^d} \mathbf{q}_{\text{ext}} \zeta + \int_{\Omega} q \zeta \circ \chi_k(t) \, dx \tag{58}$$

for all $\zeta \in W^{1,p}(\mathbb{R}^d)$. Since the right-hand side converges strongly in $W^{1,p}(\mathbb{R}^d)^*$, and since the left-hand side has a uniformly convex potential, it is standard to conclude that ϕ_k converges strongly to ϕ and that ϕ solves

$$\int_{\mathbb{R}^d} (\varepsilon_0 + \varepsilon_1 |\nabla \phi(t)|^{p-2}) \nabla \phi(t) \cdot \nabla \zeta \, dx = \int_{\mathbb{R}^d} \mathbf{q}_{\text{ext}} \zeta + \int_{\Omega} q \zeta \circ \chi(t) \, dx. \tag{59}$$

By Fubini’s theorem, we can further integrate with respect to t to obtain (34b).

The scalar-valued functions $t \mapsto \|\nabla^2 \chi_k(t) - \nabla^2 \chi(t)\|_{L^2(\Omega; \mathbb{R}^{d \times d \times d})}$, $t \mapsto \|\nabla \chi_k^{-\top}(t) - \nabla \chi^{-\top}(t)\|_{C(\bar{\Omega}; \mathbb{R}^{d \times d})}$, and $t \mapsto \|\nabla \chi_k^{-\top}(t) - \nabla \chi^{-\top}(t)\|_{C(\bar{\Omega}; \mathbb{R}^{d \times d})}$ converge to zero strongly in $L^r(I)$, and hence they also converge almost everywhere in I , that is,

$$\nabla^2 \chi_k(t) \rightarrow \nabla^2 \chi(t) \quad \text{strongly in } L^2(\Omega; \mathbb{R}^{d \times d \times d}) \quad \text{for a.a. } t \in I, \tag{60a}$$

$$\nabla \chi_k(t) \rightarrow \nabla \chi(t) \quad \text{strongly in } C(\bar{\Omega}; \mathbb{R}^{d \times d}) \quad \text{for a.a. } t \in I, \tag{60b}$$

$$\nabla \chi_k^{-\top}(t) \rightarrow \nabla \chi^{-\top}(t) \quad \text{strongly in } C(\bar{\Omega}; \mathbb{R}^{d \times d}) \quad \text{for a.a. } t \in I, \tag{60c}$$

$$\nabla \chi_k(t) \rightarrow \nabla \chi(t) \quad \text{strongly in } C(\Gamma; \mathbb{R}^{d \times d}) \quad \text{for a.a. } t \in I. \tag{60d}$$

We now examine the limit passage in the various terms in the approximate form of (34a). The limit passage in the first and third terms of the left-hand side of (34a) are immediate since the deformation appears linearly, and weak convergence suffices. The limit passage in the second term on the left-hand side of (34a) follows from the continuity properties φ' that ensue from the qualification (29a) of φ , from the convergence (52), and from Lebesgue dominated-convergence theorem. Here it is important that, even on the Galerkin level, due to [14, Theorem 3.1], we are uniformly on a sufficiently big level set of the energy $\int_{\Omega} \varphi(\nabla \chi) \, dx + \mathcal{H}(\nabla^2 \chi)$ so that we are also uniformly away of the singularity of φ at $\det \mathbf{F} = 0$, so $\varphi'(\chi_k)$ is even uniformly bounded.

We now focus our attention on the right-hand side of (34a). For the first term, we have

$$\int_{\Omega} \phi_k(t) \circ \chi_k(t) \nabla \chi_k^{-\top}(t) : \nabla(q\zeta(t)) \, dx \rightarrow \int_{\Omega} \phi(t) \circ \chi(t) \nabla \chi^{-\top}(t) : \nabla(q\zeta(t)) \, dx. \tag{61a}$$

For the second term, we observe that, since $\nabla \chi^{-T} = \left[\frac{\text{Cof}}{\det}\right](\nabla \chi)$, we have,

$$\text{div}(\nabla \chi^{-\top}) = \left[\frac{\text{Cof}}{\det}\right]'(\nabla \chi) : \nabla^2 \chi,$$

and hence, by the strong convergence of $\phi_k(t) \circ \chi_k(t)$ and $\nabla \chi_k(t)$, and the weak convergence of $\nabla^2 \chi$, we have

$$\begin{aligned} & \int_{\Omega} q \phi_k(t) \circ \chi_k(t) \text{div}(\nabla \chi_k^{-\top}(t)) \cdot \zeta(t) \, dx \\ &= \int_{\Omega} q \phi_k(t) \circ \chi_k(t) \left[\frac{\text{Cof}}{\det}\right]'(\nabla \chi_k) : \nabla^2 \chi_k(t) \cdot \zeta(t) \, dx \\ &\rightarrow \int_{\Omega} q \phi(t) \circ \chi(t) \zeta(t) \cdot \left[\frac{\text{Cof}}{\det}\right]'(\nabla \chi(t)) : \nabla^2 \chi_k(t) \, dx \\ &= \int_{\Omega} q \phi(t) \circ \chi(t) \text{div}(\nabla \chi^{-\top}(t)) \cdot \zeta(t) \, dx. \end{aligned} \tag{61b}$$

Finally,

$$\int_{\Gamma} q\phi_k(t)\circ\chi_k(t) \nabla\chi_k^{-\top}(t)\mathbf{n} \cdot \boldsymbol{\zeta} \, dS \rightarrow \int_{\Gamma} q\phi(t)\circ\chi(t) \nabla\chi^{-\top}(t)\mathbf{n} \cdot \boldsymbol{\zeta} \, dS \tag{61c}$$

for a.a. $t \in I$. Note that, in (61a), we used that $\nabla q \in L^1(\Omega; \mathbb{R}^d)$ while, in (61c), we used that $q|_{\Gamma} \in L^1(\Sigma)$ if $q \in W^{1,1}(\Omega)$, as assumed. Then we can integrate (61) over I , and prove the convergence again by using Lebesgue dominated-convergence theorem, relying on a common majorant which is even constant due to the $L^\infty(I)$ -estimates at disposal. The convergence in the approximate form of (34a) towards its limit is then proved. \square

4. Flow of a charged diffusant q . Considering the charge density not fixed with the elastic material but rather with a diffusant that can move throughout the poroelastic medium (so that the field q becomes an additional unknown) and then the stored energy $\varphi = \varphi(\nabla\chi, q)$, we can consider $\mathcal{E} = \mathcal{E}(\chi, q, \phi)$ again defined by (14), i.e.

$$\begin{aligned} \mathcal{E}(\chi, \phi, q) = & \int_{\Omega} \varphi(\nabla\chi, q) + q\phi \circ \chi \, dx \\ & + \int_{\mathbb{R}^d} \mathbf{q}_{\text{ext}}\phi - \frac{\varepsilon_0}{2}|\nabla\phi|^2 - \frac{\varepsilon_1}{p}|\nabla\phi|^p \, dx + \mathcal{H}(\nabla^2\chi). \end{aligned} \tag{62}$$

Then $\mu := \partial_q \mathcal{E}(\chi, q, \phi)$ is in the position of an *electrochemical potential*. In fact, we admit φ nonsmooth at $q = 0$, we rather write it as an inclusion, i.e.

$$\mu \in \partial_q \varphi(\nabla\chi, q) + \phi(\chi). \tag{63a}$$

The system can now be expanded also by the mass-balance equation written in the reference configuration together with the boundary conditions

$$\dot{q} - \text{div}(\mathbf{M}(x, \nabla\chi, q) \nabla\mu) = 0 \quad \text{in } \Omega \text{ (at a given time } t), \tag{63b}$$

$$(\mathbf{M}(x, \nabla\chi, q) \nabla\mu) \cdot \mathbf{n} + \alpha\mu = \alpha\mu_b, \quad \text{on } \Gamma \text{ (at a given time } t), \tag{63c}$$

where $\mathbf{M} = \mathbf{M}(x, \nabla\chi, q)$ is the mobility tensor, α is a permeability coefficient of the boundary, \mathbf{n} denotes the unit outward normal to the boundary $\Gamma = \partial\Omega$, and μ_b is a prescribed external electro-chemical potential. Under the assumption that the mobility tensor in the current configuration, namely $\mathbf{M} : \Omega \rightarrow \mathbb{R}^{d \times d}$ does not depend on \mathbf{F} , its pullback is

$$\mathbf{M}(x, \mathbf{F}, q) = \frac{(\text{Cof } \mathbf{F})^\top \mathbf{M}(x, q) \text{Cof } \mathbf{F}}{\det \mathbf{F}} \quad \text{with } x \in \Omega, \tag{63d}$$

where $\text{Cof } \mathbf{F} = (\det \mathbf{F})\mathbf{F}^{-\top}$ is the cofactor of \mathbf{F} . Here $\mathbf{M} = \mathbf{M}(x, q)$ is the material mobility tensor. In literature, this formula is often used in the isotropic case $\mathbf{M}(x, q) = m(q)\mathbb{I}$ where (63d) can easily be written by using the right Cauchy-Green tensor $\mathbf{C} = \mathbf{F}^\top \mathbf{F}$ as $\det(\mathbf{C}^{1/2})k(q)\mathbf{C}^{-1} = (\det(\mathbf{F}^\top) \det(\mathbf{F}))^{1/2}m(q)(\mathbf{F}^\top \mathbf{F})^{-1} = \det(\mathbf{F})\mathbf{F}^{-1}(k(q)\mathbb{I})\mathbf{F}^{-\top}$, cf. e.g. [8, Formula (67)] or [12, Formula (3.19)]. For a general case, see [18, Sect. 9.1].

Such system (21) with (63) is no longer conservative as diffusive processes are dissipative. The dissipation rate is $2\mathcal{R}(\chi, q, \mu) = \int_{\Omega} (\nabla\mu)^\top \mathbf{M}(x, \nabla\chi, q) \nabla\mu \, dx + \int_{\partial\Omega} \alpha\mu^2 \, dS$. Considering for a moment $\mu_b = 0$, analogous to (20), the enhanced

system has the abstract structure

$$\partial_{\dot{\chi}} \mathcal{F} \ddot{\chi} + \partial_{\chi} \mathcal{E}(\chi, q, \phi) = \mathcal{F}(t), \tag{64a}$$

$$\partial_{\dot{q}} \mathcal{R}^*(\chi, q) \dot{q} + \partial_q \mathcal{E}(\chi, q, \phi) = 0, \tag{64b}$$

$$\partial_{\phi} \mathcal{E}(\chi, q, \phi) = 0, \tag{64c}$$

where the potential of dissipative forces expressed in terms of the rate \dot{q} , i.e.

$$\mathcal{R}(\chi, q, \dot{q}) = \int_{\Omega} \frac{1}{2} |\mathbf{M}(\nabla \chi, q)^{1/2} \nabla \Delta_{\mathbf{M}(\nabla \chi, q), \alpha}^{-1} \dot{q}|^2 dx + \int_{\partial \Omega} \frac{\alpha}{2} (\Delta_{\mathbf{M}(\nabla \chi, q), \alpha}^{-1} \dot{q})^2 dS, \tag{65}$$

is quadratic in term of the rate \dot{q} with $\Delta_{\mathbf{M}, \alpha}^{-1} : \xi \mapsto q$ denoting the linear operator which assigns ξ the (weak) solution q to the boundary-value problem $\xi = -\text{div}(\mathbf{M} \nabla q)$ with the boundary condition $(\mathbf{M} \nabla q) \cdot n + \alpha q = 0$. In fact, $\mathcal{R}^*(\chi, \cdot)$ is the convex conjugate functional to the $\mathcal{R}(\chi, \cdot)$. A general μ_b would give rise to a non-zero right-hand side in (64b), cf. also [24].

The energy balance on a time interval $[0, t]$ can be revealed by testing the particular equations (64) respectively by $\dot{\chi}$, \dot{q} , and ϕ :

$$\begin{aligned} & \underbrace{\mathcal{F}(\dot{\chi}(t)) + \mathcal{E}(\chi(t), q(t), \phi(t))}_{\text{kinetic and stored energy at time } t} + 2 \underbrace{\int_0^t \mathcal{R}(\chi, \mu) dt}_{\text{energy dissipated on the time interval } [0, t]} \\ &= \underbrace{\mathcal{F}(\dot{\chi}(0)) + \mathcal{E}(\chi(0), q(0), \phi(0))}_{\text{kinetic and stored energy at the initial time 0}} + \underbrace{\int_0^t \left(\langle \mathcal{F}, (\dot{\chi}, \dot{\phi}) \rangle + \int_{\Gamma} \mu_{\text{ext}} \mu dS \right) dt}_{\text{work done by external mechano-chemical loading}} \end{aligned} \tag{66}$$

with $\mu = \partial_q \mathcal{E}(\chi, q, \phi)$, cf. (20) where $\mathcal{R} = 0$.

A more specific example of free energy is the celebrated Biot model [3] formulated (analogously as at small strains) here at large strains, cf. also e.g. [8], resulting into a potential

$$\varphi(\mathbf{F}, q) = \varphi_s(\mathbf{F}) + \frac{1}{2} M_B (q - q_E \beta (1 - \det \mathbf{F}))^2 + \begin{cases} \kappa q (\ln(q/q_E) - 1) & \text{for } q > 0, \\ 0 & \text{for } q = 0, \\ +\infty & \text{for } q < 0, \end{cases} \tag{67}$$

where q is a mass concentration of a diffusant and q_E is an equilibrium concentration, $M_B > 0$ the so-called Biot modulus, $\beta \geq 0$ the Biot coefficient, and $\kappa > 0$ a coefficient. Let us notice the singularity of $\partial_q \varphi(\mathbf{F}, q)$ at $q = 0$, which ensures non-negativity of the diffusant concentration q . Here φ_s plays the role of φ in Sections 2 and 3.

To facilitate the analysis, we neglect the mass density of the diffusant, cf. Remark 4 below.

Let us now briefly present the analysis if the diffusion (63) is involved. The definition of the weak solution must be formulated carefully if ϱ has a singularity as in (67), combining the concept of the variational solution for (63a) and the conventional weak solution for (63b,c) with the initial condition for q :

Definition 4.1 (Weak solution to the problem (21) with diffusion (63)). A quadruple $(\chi, \phi, q, \mu) \in C_w(I; H^{2+\gamma}(\Omega; \mathbb{R}^d)) \times L^\infty(I; W^{1,p}(\mathbb{R}^d)) \times C_w(I; L^2(\Omega)) \times$

$L^2(I; H^1(\Omega))$ is a weak solution to (21) and (63) with (22)–(23) and the initial condition $q|_{t=0} = q_0$ if $\mathcal{H}(\nabla\chi) \in L^\infty(I)$, $\dot{\chi} \in C_w(I; L^2(\Omega; \mathbb{R}^d))$, and $\dot{q} \in L^2(I; H^1(\Omega)^*)$, and if (34a) with $\partial_{\mathbf{F}}\varphi(\nabla\chi, q)$ instead of $\partial\varphi(\nabla\chi)$ holds together with the initial condition $\dot{\zeta}(0) = \mathbf{v}_0$, and if, for every $w \in H^1(Q; \mathbb{R}^d)$ satisfying $w(T) = 0$, it holds

$$\int_Q M(\nabla\chi) \nabla\mu \cdot \nabla w - q\dot{w} \, dxdt = \int_\Omega q_0 w(0) \, dx \tag{68}$$

with μ satisfying the variational inequality

$$\int_\Omega \varphi(\nabla\chi(t), q(t)) \, dx \leq \int_\Omega \varphi(\nabla\chi(t), \tilde{q}) + (\mu(t) - \Phi(t, \chi(t)))(\tilde{q} - q(t)) \, dx \tag{69}$$

for a.a. $t \in I$ and for all $\tilde{q} \in L^2(\Omega)$, and if also (34b) holds.

Beside (35), we now use the Galerkin approximation also for (63) with finite-dimensional subspaces of $H^1(\Omega)$. It is important (and we can assume it without loss of generality) that both (63a) and (63b) use the same finite-dimensional subspaces of $H^1(\Omega)$, which facilitates the cross-test of (63a) by \dot{q} and of (63b) by μ .

Again, we may assume that the initial conditions χ_0 , \mathbf{v}_0 , and q_0 belong to all these subspaces without loss of generality. Let us denote the approximate solution thus created by $(\chi_k, q_k, \mu_k, \phi_k)$. It is important to use the same subspace for discretisation of both equations in (63) to allow the cross-test of (63a) by \dot{q}_k and of (63b) by μ_k .

Proposition 2 (Weak solution to (21) with (63)). *Let $\varphi : \text{SL}^+(d) \times \mathbb{R} \rightarrow \mathbb{R}$ be smooth, $\varphi(\cdot, q) : \text{SL}^+(d) \rightarrow \mathbb{R}$ satisfy (29) uniformly for $q \in \mathbb{R}$, $\varphi(\mathbf{F}, \cdot)$ be uniformly convex (uniformly also with respect to \mathbf{F}) with $\partial_{\mathbf{F}q}^2\varphi : \text{GL}^+(d) \times \mathbb{R} \rightarrow \mathbb{R}^{d \times d} \times \mathbb{R}$ and $1/\partial_{qq}^2\varphi : \text{GL}^+(d) \times \mathbb{R} \rightarrow \mathbb{R}^{d \times d} \times \mathbb{R}^+$ be continuous when extended by zero for $q < 0$, and $|\partial_{\mathbf{F}q}^2\varphi(\mathbf{F}, q)| \leq C(\mathbf{F})(1 + |q|)$ with some $C : \mathbb{R}^{d \times d} \rightarrow \mathbb{R}^+$ continuous. Let also (30) and (31b) holds. Moreover, let $\chi_0 \in H^{2+\gamma}(\Omega; \mathbb{R}^d)$ and $\mathbf{v}_0 \in L^2(\Omega; \mathbb{R}^d)$ and $q_0 \in L^2(\Omega)$ with $\varphi(\nabla\chi_0, q_0) \in L^1(\Omega)$. Then, together with the a-priori estimates (41), also*

$$\|q_k\|_{L^2(I; W^{1,1}(\Omega)) \cap L^\infty(I; L^2(\Omega)) \cap H^1(I; H^1(\Omega)^*)} \leq C \quad \text{and} \tag{70a}$$

$$\|\mu_k\|_{L^2(I; H^1(\Omega))} \leq C. \tag{70b}$$

There is a subsequence of $\{(\chi_k, q_k, \mu_k, \phi_k)\}_{k \in \mathbb{N}}$ converging weakly* in the topologies indicated in (41) and the limit of any such subsequence is the weak solution to (21) with (63) with the initial conditions $\chi(0) = \chi_0$, $\dot{\chi} = \mathbf{v}_0$, and $q(0) = q_0$.

Proof. We perform the energetic test of the system (21) with (64) by successively $\dot{\chi}_k$, $\dot{\phi}_k$, \dot{q}_k , and μ_k .

By a manipulation as in (43)–(50) extended now by the dissipative diffusion, we obtain

$$\begin{aligned} & \frac{d}{dt} \left(\int_\Omega \frac{\rho}{2} |\dot{\chi}_k|^2 + \varphi(\nabla\chi_k, q_k) \, dx + \int_{\mathbb{R}^d} \frac{\varepsilon_0}{2} |\nabla\phi_k|^2 + \frac{\varepsilon_1}{p'} |\nabla\phi_k|^p \, dx + \mathcal{H}(\nabla^2\chi_k) \right) \\ & \quad + \int_\Omega M(\nabla\chi_k) \nabla\mu_k \cdot \nabla\mu_k \, dx + \int_\Gamma \alpha\mu_k^2 \, dS \\ & \quad = \int_\Omega \mathbf{f} \cdot \dot{\chi}_k \, dx + \int_\Gamma \mathbf{g} \cdot \dot{\chi}_k + \alpha\mu_b\mu \, dS + \int_{\mathbb{R}^d} \dot{\mathbf{q}}_{\text{ext}}\phi_k \, dx. \end{aligned} \tag{71}$$

From this and also by comparison of \dot{q}_k , we obtain the a-priori estimates (41) and, except the $L^2(H^1)$ -estimate of q_k , also (70). More in detail, the $L^2(I; H^1(\Omega)^*)$ -estimate of \dot{q}_k in (70) is meant for a suitable Hahn-Banach extension of the functional \dot{q}_k defined originally only on the functions valued on the finite-dimensional subspace used for the k -th Galerkin approximation.

As q occurs nonlinearly in the energy φ , we need to prove its strong convergence. Even more, the limit passage in (61) modified for $q_k \rightarrow q$ needs the strong convergence of $q_k(t)$ in $W^{1,1}(\Omega)$ because, from a mere weak convergence of ∇q_k , we could not inherit the convergence for a.a. t needed in (61). As we do not use the Cahn-Hilliard model, we do not have ∇q directly estimated, but we can rely on the uniform convexity of $\varphi(\mathbf{F}, \cdot)$, as indeed e.g. in the Biot model (67). Then the estimate of ∇q can be obtained by applying ∇ -operator to (63a) to see $\nabla \mu = \partial_{\mathbf{F}q}^2 \varphi(\mathbf{F}, q) \nabla \mathbf{F} + \partial_{qq}^2 \varphi(\mathbf{F}, q) \nabla q$, so that we can estimate

$$\nabla q_k = \frac{\nabla \mu_k - \partial_{\mathbf{F}q}^2 \varphi(\nabla \chi_k, q_k) \nabla^2 \chi_k}{\partial_{qq}^2 \varphi(\nabla \chi_k, q_k)}. \tag{72}$$

provided we assume still the uniform convexity of $\varphi(\mathbf{F}, \cdot)$, as indeed e.g. in the Biot model (67). This gives the $L^2(H^1)$ -estimate of q_k in (70).

Now we select the weakly* converging subsequence in the topologies indicated in (41) and (70). Furthermore, we prove $\nabla \mu_k \rightarrow \nabla \mu$ strongly in $L^2(Q; \mathbb{R}^d)$. This can be seen from the initial-boundary-value problem (63b,c) when taking into account the uniform positive-definiteness of the pulled-back mobility matrix (63d). We have also $\nabla^2 \chi_k \rightarrow \nabla^2 \chi$ strongly in $L^2(Q; \mathbb{R}^{d \times d \times d})$ and $q_k \rightarrow q$ strongly in $L^2(Q)$. In particular, selecting possibly another subsequence, we have $\nabla \mu_k \rightarrow \nabla \mu$ strongly in $L^2(\Omega; \mathbb{R}^d)$ and $\nabla^2 \chi_k(t) \rightarrow \nabla^2 \chi(t)$ strongly in $L^2(\Omega; \mathbb{R}^{d \times d \times d})$ and also $q_k(t) \rightarrow q(t)$ strongly in $L^2(\Omega)$, from (72) we can see that $\nabla q_k(t) \rightarrow \nabla q(t)$ strongly in $L^1(Q; \mathbb{R}^d)$. Here the continuity of $\partial_{\mathbf{F}q}^2 \varphi$ and of $1/\partial_{qq}^2 \varphi$ has been used.

Note also that the arguments (58) modifies because the last integral in (58) is now $\int_{\Omega} q_k(\Phi_k \circ \chi_k) - q_j(\Phi_j \circ \chi_j) \, dx$ and it again converges to 0 when $k, j \rightarrow \infty$.

The limit passage in

$$\int_{\Omega} \varphi(\nabla \chi_k(t), q_k(t)) \, dx \leq \int_{\Omega} \varphi(\nabla \chi_k(t), \tilde{q}) + (\mu_k(t) - \Phi_k(t, \chi_k(t)))(\tilde{q} - q_k(t)) \, dx$$

towards (69) is then easy by lower semicontinuity or continuity. □

Remark 1 (*Maxwell stress*). The term $q \nabla \phi$ on the left-hand side of the pointwise force balance (21a) is the opposite of the electrostatic body-force referential (Lagrangian) density. Its spatial (Eulerian) density can be written as the divergence of a tensor field, which is then recognized as the Maxwell stress. We illustrate this fact by a formal calculation, performed under the assumption that χ is invertible and that the external charge density q_{ext} vanishes. As a start, we test (21a) by a virtual velocity ζ which is smooth and vanishes in a neighborhood of $\partial \Omega$, and then we perform an integration over Ω . In the resulting equation we focus our attention on the term

$$W(\zeta) := \int_{\Omega} q(\nabla \phi \circ \chi) \cdot \zeta \, dx = \int_{\Omega} q(\nabla \phi \cdot (\zeta \circ \chi^{-1})) \circ \chi \, dx,$$

which we interpret as the virtual work performed by the referential electrostatic body force. Further, we use the weak form (34b) of the nonlinear Poisson equation

(21b), substituting $\nabla\phi \cdot (\zeta \circ \chi^{-1})$ in place of the test field ζ . By doing so we obtain

$$\begin{aligned} W(\zeta) &= \int_{\chi(\Omega)} (\varepsilon_0 + \varepsilon_1 |\nabla\phi|^{p-2}) \nabla\phi \cdot \nabla(\nabla\phi \cdot (\zeta \circ \chi^{-1})) \\ &= \int_{\chi(\Omega)} (\varepsilon_0 + \varepsilon_1 |\nabla\phi|^{p-2}) \nabla\phi \cdot (\nabla\nabla\phi (\zeta \circ \chi^{-1}) + \nabla(\zeta \circ \chi^{-1})^\top \nabla\phi) \end{aligned}$$

Then, we recall that in our regularized model the spatial electrostatic energy density is $e(\nabla\phi)$, where $e(\mathbf{v}) = \frac{\varepsilon_0}{2} |\mathbf{v}|^2 + \frac{\varepsilon_1}{p} |\mathbf{v}|^p$. Thus, on introducing the electric field $\mathbf{e} = -\nabla\phi$, the electric displacement $\mathbf{d} = e'(\mathbf{e}) = (\varepsilon_0 + \varepsilon_1 |\nabla\phi|^{p-2}) \mathbf{e}$, and the *spatial test velocity* $\zeta = \zeta \circ \chi^{-1}$, we can write

$$\begin{aligned} W(\zeta) &= \int_{\chi(\Omega)} e'(\nabla\mathbf{e}) \cdot (\nabla\mathbf{e} \zeta) + \mathbf{d} \cdot \nabla\zeta^\top \mathbf{e} \\ &= \int_{\chi(\Omega)} \nabla e(\mathbf{e}) \cdot \zeta + (\mathbf{e} \otimes \mathbf{d}) \cdot \nabla\zeta = \int_{\chi(\Omega)} ((\mathbf{e} \otimes \mathbf{d}) - e(\mathbf{e})\mathbf{I}) : \nabla\zeta. \end{aligned}$$

We recognize the spatial tensor field $\mathbf{M} = (\mathbf{e} \otimes \mathbf{d}) - e(\mathbf{e})\mathbf{I}$ to be the so-called Maxwell stress.

Remark 2 (*Darcy or Fick diffusion in the Biot poroelastic model*). For a special choice of the Biot model (67) for $\varphi = \varphi(\nabla\chi, q)$ in (62), we obtain the stress and the chemical potential as

$$\mathbf{S} = \varphi'_s(\mathbf{F}) + \beta q_E p \text{Cof } \mathbf{F} \quad \text{with} \quad p = M_B (q - q_E \beta (1 - \det \mathbf{F})) \quad \text{and} \quad (73a)$$

$$\mu \in \partial_q \varphi_s(\mathbf{F}, q) = \begin{cases} p + \kappa \ln(q/q_E) & \text{for } q > 0. \\ \emptyset & \text{for } q = 0, \end{cases} \quad (73b)$$

For a standard choice $\mathbf{M}(q) = q\mathbf{M}_0$ in (63d), the flux $j = -\mathbf{M}(\mathbf{F}, q) \nabla\mu$ results to

$$j = \underbrace{-q\mathbf{M}_0(\mathbf{F}) \nabla p}_{\text{Darcy law}} - \underbrace{\kappa\mathbf{M}_0(\mathbf{F}) \nabla q}_{\text{Fick law}} \quad \text{with} \quad \mathbf{M}_0(\mathbf{F}) = \frac{(\text{Cof } \mathbf{F})^\top \mathbf{M}_0 \text{Cof } \mathbf{F}}{\det \mathbf{F}} \quad (74)$$

provided $q > 0$. Depending on the coefficient $\kappa > 0$, either Darcy’s mechanism or the Fick’s one may dominate. In particular, we identified the “diffusant pressure” p in (73a) which governs the Darcy law. For the Biot model under large strains, we also refer to [4] and, at small strains, to [25]. Counting the charges as occurring in the electrochemical potential (63a), the Darcy/Fick law (74) enhances still as

$$j = -q\mathbf{M}_0(\mathbf{F}) \nabla p - \kappa\mathbf{M}_0(\mathbf{F}) \nabla q - q\mathbf{M}_0(\mathbf{F}) \nabla\phi(\chi), \quad (75)$$

where the last term is a so-called drift current. We thus can see a drift-diffusion model (as used e.g. in monopolar semiconductors) combined with the Darcy flow due to mechanical pressure gradient.

5. Final remarks: Generalizations or modifications. Let us end this article by several remarks outlining possible generalizations or modifications and, in most of cases, serious difficulties related with them.

Remark 3 (*Multi-component flow in charge poroelastic solid*). The models from Sections 2 and 4 can be easily merged in the sense that the fixed q from Section 2 is to be interpreted as the charge density of dopants in the poroelastic solid while the varying q from Section 4 is the charge density of diffusant. This may model elastic negatively-charged porous polymers with hydrogen cations (protons) in so-called polymer electrolytes as the central layer of PEM fuel cells [22] or unipolar (elastic)

semiconductor devices as field-effect transistors FETs or Gunn’s diodes. Even, this latter q may be vector-valued if there are more than one diffusant. Even chemical reactions between particular components of the diffusant can be involved. Then one should rather speak about concentrations c_i , $i = 1, \dots, n$, of the charged diffusants with the specific charges q_i . Denoting the charge of the fixed dopant as q_0 , the system (21)–(63a,b) can be generalized as:

$$\varrho \ddot{\chi} - \operatorname{div}(\varphi'(\nabla \chi) - \operatorname{div} \mathfrak{H}(\nabla^2 \chi)) + q \nabla \phi(\chi) = f \tag{76a}$$

$$\operatorname{div}((\varepsilon_0 + \varepsilon_1 |\nabla \phi|^{p-2}) \nabla \phi) + \sum_{x \in \tilde{\chi}(\cdot, t)} \frac{q(x) + \sum_{i=1}^n c_i(x) q_i(x)}{\det(\nabla \chi(t, x))} + \mathbf{q}_{\text{ext}}(t, \cdot) = 0 \tag{76b}$$

$$\dot{c}_i - \operatorname{div}(\mathbf{M}(x, \nabla \chi, c_1, \dots, c_n) \nabla \mu) = r_i(c_1, \dots, c_n) \quad (i = 1, \dots, n), \tag{76c}$$

$$\mu_i \in \partial_{c_i} \varphi(\nabla \chi, c_1, \dots, c_n) + \phi(\chi) \quad (i = 1, \dots, n), \tag{76d}$$

with $r_i(q)$ the chemical-reaction rate of the i -th constituent and \mathbf{M} is now valued in $\mathbb{R}^{d \times d \times n}$, caring also about cross-diffusion effects. The specific applications may cover hydrogen oxidation and the oxygen reduction reactions in porous electrodes (i.e. cathode/anode layers) of PEM fuel cells [22] influenced by mechanical loading [2, 19] or in bipolar doped semiconductor devices (diodes, transistors, thyristors) with chemically reacting (through generation/recombination mechanisms) electron and hole charges under mechanical load, or Li-cations and electrons in porous batteries, etc.

Remark 4 (*Inertia of the diffusant*). If the mass density of the diffusant denoted here by ϱ_1 is not negligible, one should rather consider the overall mass density as $\varrho = \varrho_0 + q\varrho_1$ with ϱ_0 now standing for the mass density of the poroelastic solid itself (e.g. a polymeric matrix or a porous rock). The energetic test of the inertial term $\varrho \ddot{\chi}$ by $\dot{\chi}$ would then lead to $\frac{\partial}{\partial t} \frac{1}{2} \varrho |\dot{\chi}|^2 - \frac{1}{2} \dot{q} \varrho_1 |\dot{\chi}|^2$. The newly arising term $\frac{1}{2} \dot{q} \varrho_1 |\dot{\chi}|^2$, which would have to be estimated “on the right-hand side, seems to bring serious difficulties even for the a-priori estimation. In the weak formulation (34a), we would obtain still the term $\int_Q \dot{\varrho} \dot{\chi} \cdot v \, dx dt$ which requires to introduce a viscosity in (63a) to control $\dot{\varrho} = (\varrho_s + q) \dot{q}$ in $L^2(Q)$. For analysis, (63a) is to be still augmented by the term $-\operatorname{div}(\tau_r |\nabla \dot{q}|^{p-2} \nabla \dot{q})$ with $p > 3$ and with $\tau_r > 0$ a relaxation time; such a gradient-viscous Cahn-Hilliard model was suggested in [13], and then, assuming a suitable regularization of (67) to guarantee $|\partial_q \varphi(\mathbf{F}, q)| \leq C(1+|q|)$, e.g. $\det \mathbf{F} / (1 + \varepsilon \det \mathbf{F})$ instead of $\det \mathbf{F}$ for small $\varepsilon > 0$, we can test separately (64b) by \dot{q} to estimate \dot{q} in $L^p(I; W^{1,p}(\Omega))$ and then, using $W^{1,p}(\Omega) \subset L^\infty(\Omega)$, treat $\int_0^t \int_\Omega \dot{\varrho} |\dot{\chi}|^2 \, dx dt$ by Gronwall inequality. Then (65) is to be augmented by some viscosity contribution like $\tau_r |\nabla \dot{q}|^p / p$. Also the “dual” estimate of $\varrho \ddot{\chi}$ seems problematic.

Remark 5 (*Attractive monopolar interactions*). Another prominent example of a monopolar interaction is gravitation. Then q is mass density and ϕ is the gravitational field. The essential difference is that the gravitational constant occurs instead of ε in (1) and then also in (21b) and (9) with a negative sign. The coercivity of the static stored energy is not automatic and, roughly speaking needs sufficiently small total mass of a medium sufficiently elastically tough. On top of it, $\varepsilon_1 = 0$ is the only reasonable choice in such gravitation interaction, which is not covered by the proof of Proposition 1.

Remark 6 (*Dipolar long-range interactions*). Considering a vector-valued density $\vec{q} : \Omega \rightarrow \mathbb{R}^m$ and $\vec{q}_{\text{ext}}(\cdot, t) : \mathbb{R}^d \rightarrow \mathbb{R}^m$ instead of just scalar valued does not change (6) and (6), but the stored energy (14) is to be modified as:

$$\mathcal{E}(\chi, \phi) = \int_{\Omega} \varphi(\nabla \chi) - \vec{q} \cdot \nabla \phi(\chi) \, dx + \int_{\mathbb{R}^d} \frac{\kappa}{2} |\nabla \phi(x)|^2 \, dx + \mathcal{H}(\nabla^2 \chi).$$

Instead of (21), the Hamiltonian variational principle then modifies the system (21) as

$$\rho \ddot{\chi} - \operatorname{div} \mathbf{S} = \mathbf{f} + [(\nabla^2 \phi) \circ \chi] \vec{q} \quad \text{with} \quad \mathbf{S} = \varphi'(\nabla \chi) - \operatorname{Div}(\mathfrak{H} \nabla^2 \chi) \quad (77a)$$

$$\operatorname{div}(\varepsilon \nabla \phi) = \operatorname{div} \left(\sum_{x \in \chi^{-1}(\cdot, t)} \frac{\vec{q}(x)}{\det(\nabla \chi(x, t))} + \vec{q}_{\text{ext}}(\cdot, t) \right) \quad \text{for a.a. } t \in I. \quad (77b)$$

The interpretation is of ϕ and \vec{q} and of κ is the potential of magnetic field and magnetization and permeability in elastic ferromagnets, respectively, or alternatively electrostatic field and polarization and permittivity in elastic piezoelectric materials with spontaneous polarization. The analysis is, however, even more problematic comparing to the repulsive monopolar case due to less regularity of the Poisson equation (77b) which has the divergence of an L^1 -function in the right-hand side. Actually, this difficulty is not seen in the a-priori estimation strategy (71) as well as in the limit passage in the Poisson equation (77b) written in the form (34b). Yet, the difficulty occurs in the term $(\nabla^2 \phi) \circ \chi$ in the right-hand side of (77a) because $\nabla^2 \phi$ hardly can be continuous and the composition with χ even does not need to be measurable.

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E-mail address: tomas.roubicek@mff.cuni.cz

E-mail address: giuseppe.tomassetti@uniroma3.it