

A convective model for poro-elastodynamics with damage and fluid flow towards Earth lithosphere modelling.

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Abstract Devised towards geophysical applications for various processes in the lithosphere or the crust, a model of poro-elastodynamics with inelastic strains and other internal variables like damage (aging) and porosity as well as with diffusion of water is formulated fully in the Eulerian setting. Concepts of gradient of the total strain rate as well as the additive splitting of the total strain rate are used while eliminating the displacement from the formulation. It relies on that the elastic strain is small while only the inelastic and the total strains can be large. The energetics behind this model is derived and used for analysis as far as the existence of global weak energy-conserving solutions concerns. By this way, the model of V. Lyakhovskiy et al. [20,22] is completed to make it mechanically consistent and amenable for analysis.

Keywords Inelasticity · damage mechanics · diffusion · Eulerian description · Korteweg stress · weak solutions

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1 Introduction

Geophysical models of the solid Earth (i.e. particularly the lithosphere and the crust) are extremely challenging applications of continuum mechanics. Such models should capture a lot of phenomena on various time-space scales and usually are focused on only specific aspects, cf. [1, Fig. 1] for the spatiotemporal scales relevant for earthquakes and fault dynamics. On short time scales, fast rupture of lithospheric faults, tectonic earthquakes, and seismic waves are most prominent phenomena. On large time scales, aseismic creep, healing of damaged faults, and water (or sometimes oil) transport in porous rocks are dominant effects to capture. The water transport processes are intimately coupled with mechanical properties and possibly also with evolution of porosity and of damage (called also aging in geophysical applications). Other effects would be heat production and transfer, magnetism, or volcanism, but we will not consider them in the model formulated here.

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Although the full model should be formulated at large strains as in [36], geophysical applications in solid parts of the Earth (mainly the crust and the lithospheric parts of the mantle) are formulated at small strains, which can also be more efficiently implemented on computers. Even, mostly seismic sources (tectonic earthquakes by fast ruptures of lithospheric faults) are separated from seismic wave propagation in most of geophysical simulations, although physically these two processes are obviously coupled as also captured in the model presented here. Simultaneously with this small elastic strain assumption which is well relevant in all processes in the lithosphere, there might be a large inelastic strain accumulated during slow tectonic processes on the mentioned large-time scales. Simultaneously, we will consider inertia so that seismic waves typically emitted during fast damage and subsequent inelastic shift during earthquakes are not excluded from the model.

Large inelastic and total strains lead in general also to large displacements. Then the usual dilemma between the Lagrangian and the Eulerian description arises. In contrast to the standard choice in solid mechanics, we will use the Eulerian description like suggested essentially in [20,22]. Then, all time derivatives in the model should be convective, i.e. the material derivatives. As a consequence, in particular, the Korteweg-like stresses arise from the gradients of internal variables and the inertial forces need careful formulation and treatment like in fluid mechanics of so-called quasi-incompressible fluids, cf. [38,39], refined in the context of elastic “semi-compressible” fluids in the consistent Eulerian description in [33, Sect.5].

As e.g. in [21,22,31], we use the Green-Naghdi [10] additive splitting of the total strain but do not assume the inelastic strain to be small. In contrast to [31], we formulate the model fully in the Eulerian setting, so that all time-derivatives are convective (= material). Thus, in contrast to [22], where the structural stress is incomplete and no energy balance is thus achieved, we have the correct energy balance rigorously at disposal. An important attribute is that, like in [22], we formulate the model not in terms of displacements but rather in terms of velocities and strains. We admit stored energies which are nonconvex in the elastic strain like devised in [23] to model unstable response of damaged rocks and used e.g. in numerous geophysical articles as e.g. [1,9,20,22], and simultaneously do not use a total-strain gradient (which would not be physically consistent) but only a total-strain-rate gradient.

The goal of this article is to devise the models from [20,22] correctly to respect energy balance and, thus, to allow for rigorous analysis. Also [31], where the inertial term was not formulated in the convective way and thus the energy balance contained some nonphysical term and where (rather for analytical reasons but not physically motivated) the gradient of the total strain was used, will thus be improved. The fluid flow in poroelastic medium, like devised in [21] without damage gradient, will be consistently incorporated into the model, too.

2 The poro-elastodynamical model

We consider a continuum whose motion takes place in a fixed region Ω of space. We denote by x and t the typical point of Ω and the typical time. The kinematical ingredients (basic variables of the model) are

\mathbf{v}	velocity (valued in \mathbb{R}^d),
\mathbf{E}	elastic strain (valued in $\mathbb{R}_{\text{sym}}^{d \times d}$),
\mathbf{I}	inelastic (plastic-like) strain (valued in $\mathbb{R}_{\text{sym}}^{d \times d}$),
α	other internal variables (as damage, breakage, and/or porosity, valued in \mathbb{R}^ℓ),
χ	water (or oil) content (scalar valued),

with $\mathbb{R}_{\text{sym}}^{d \times d}$ denoting the set of symmetric $d \times d$ -matrices. In addition, we shall use the auxiliary variable μ which will be in a position of a chemical potential, having here a concrete meaning of the so-called pore pressure. The inelastic strain can incorporate a creep strain to describe Maxwellian rheology or plastic strain to describe activated slip processes which develop, for example, during earthquakes.

We investigate the following system of partial differential equations/inclusions:

$$\rho \frac{D\mathbf{v}}{Dt} = \operatorname{div}(\partial_{\mathbf{E}}\varphi(\mathbf{E}, \alpha, \chi) + k_v \mathbf{e}(\mathbf{v}) + \mathbf{S}_{\text{str}}) + \mathbf{f} - \frac{\rho}{2}(\operatorname{div} \mathbf{v})\mathbf{v}, \quad (2.1a)$$

$$\frac{D\mathbf{E}}{Dt} = \mathbf{e}(\mathbf{v}) - \frac{D\mathbf{II}}{Dt} + k_e \Delta \partial_{\mathbf{E}}\varphi(\mathbf{E}, \alpha, \chi) \quad \text{with} \quad \mathbf{e}(\mathbf{v}) := \operatorname{sym} \nabla \mathbf{v}, \quad (2.1b)$$

$$\partial_{\frac{D\mathbf{II}}{Dt}} \zeta\left(\alpha, \chi; \frac{D\mathbf{II}}{Dt}, \frac{D\alpha}{Dt}\right) - \partial_{\mathbf{E}}\varphi(\mathbf{E}, \alpha, \chi) \ni k_p \Delta \mathbf{II}, \quad (2.1c)$$

$$\partial_{\frac{D\alpha}{Dt}} \zeta\left(\alpha, \chi; \frac{D\mathbf{II}}{Dt}, \frac{D\alpha}{Dt}\right) + \partial_{\alpha}\varphi(\mathbf{E}, \alpha, \chi) \ni k_a \Delta \alpha, \quad (2.1d)$$

$$\frac{D\chi}{Dt} = \operatorname{div}(\mathbb{M}(\alpha, \chi)\nabla\mu) \quad \text{with} \quad \mu = \partial_{\chi}\varphi(\mathbf{E}, \alpha, \chi), \quad (2.1e)$$

where we use the conventional notation

$$\frac{D(\cdot)}{Dt} = \left[\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right](\cdot) \quad (2.2)$$

to denote the material derivative with respect to time and where “ ∂ ” denotes the partial derivatives or, in (2.1c,d), the convex subdifferential to allow for nonsmoothness of the dissipation potential $\zeta(\alpha, \chi; \cdot, \cdot)$ at zero rates to model activated processes in inelastic strain and damage/porosity evolution. Here $\rho > 0$ is a reference mass density and $\varphi(\mathbf{E}, \alpha, \chi)$ is the free-energy density. Moreover, \mathbf{S}_{str} in (2.1a) is the *structural stress* (called also Korteweg’s [15] or Ericksen’s [8] stress) given here as

$$\mathbf{S}_{\text{str}} = k_p \nabla \mathbf{II} \boxtimes \nabla \mathbf{II} + k_a \nabla \alpha \boxtimes \nabla \alpha - \left(\varphi(\mathbf{E}, \alpha, \chi) + \frac{k_p}{2} |\nabla \mathbf{II}|^2 + \frac{k_a}{2} |\nabla \alpha|^2 \right) \mathbf{I}. \quad (2.3)$$

The constants k_p and k_a appearing in (2.1c,d) and (2.3) determine the length-scale of the inelastic strain and of the other internal variables. In fact, k_a can rather be a matrix, expressing different length-scale for particular internal variables and possible cross-effects. The coefficient k_v in (2.1a) corresponds to the Kelvin-Voigt rheology, but when combined with a Maxwell rheology which may be governed by (2.1c), we actually obtain the Jeffreys’ rheology, as used e.g. in [22]. Moreover, $\nabla \mathbf{II} \boxtimes \nabla \mathbf{II} = \sum_{i,j=1}^d \nabla(\mathbf{II})_{ij} \otimes \nabla(\mathbf{II})_{ij}$, i.e. component-wise $[\nabla \mathbf{II} \boxtimes \nabla \mathbf{II}]_{ij} = \sum_{k,l=1}^d \frac{\partial}{\partial x_i}(\mathbf{II})_{kl} \frac{\partial}{\partial x_j}(\mathbf{II})_{kl}$, and similarly $\nabla \alpha \boxtimes \nabla \alpha = \sum_{i=1}^{\ell} \nabla \alpha_i \otimes \nabla \alpha_i$, whereas \mathbf{I} is the identity matrix. Furthermore, $\zeta(\alpha, \chi; \frac{D\mathbf{II}}{Dt}, \frac{D\alpha}{Dt})$ in (2.1c,d) is the dissipation potential in general nonsmooth at $\frac{D\mathbf{II}}{Dt} = 0$ and $\frac{D\alpha}{Dt} = 0$.

The equation (2.1e) is (the convective variant of) the standard Fick-type diffusion driven by the gradient of the chemical potential μ with $\mathbb{M} = \mathbb{M}(\alpha, \chi)$ being a positive-definite mobility matrix. The structural force, i.e. the last term in (2.1a), was proposed by R. Temam [38], cf. also [39, Ch. III, § 8]. Beside balancing energetics, this force vanishes in the incompressible limit, which was the motivation of [38]. The calculations we perform below provide a justification of this term. The decomposition (2.1b) is legitimate in some special (particular in stratified situations), cf. Remark 2.3, below, while the diffusion in (2.1b) is discussed in Remark 2.4. The structural stresses (2.3) are usually negligible but are important, beside balancing energetics, “in narrow zones with high damage gradients or damage fronts separating between areas with intact and highly damaged material”, as claimed in [20].

We have to complete the system (2.1)–(2.3) by suitable boundary conditions, say

$$\mathbf{v} \cdot \mathbf{n} = 0, \quad (2.4a)$$

$$\left((\partial_{\mathbf{E}}\varphi(\mathbf{E}, \alpha, \chi) + k_v \mathbf{e}(\mathbf{v}) + \mathbf{S}_{\text{str}}) \mathbf{n} \right)_{\mathbf{t}} + \gamma \mathbf{v}_{\mathbf{t}} = \mathbf{g}_{\mathbf{t}}, \quad (2.4b)$$

$$(\mathbf{n} \cdot \nabla) \partial_{\mathbf{E}}\varphi(\mathbf{E}, \alpha, \chi) = 0, \quad (\mathbf{n} \cdot \nabla) \mathbf{II} = 0, \quad \nabla \alpha \cdot \mathbf{n} = 0, \quad (2.4c)$$

$$\mathbb{M}(\alpha, \chi) : (\nabla \mu \otimes \mathbf{n}) = h, \quad (2.4d)$$

where $(\cdot)_{\mathbf{t}}$ denotes the tangential component of a vector, i.e. e.g. $\mathbf{v}_{\mathbf{t}} = \mathbf{v} - (\mathbf{v} \cdot \mathbf{n})\mathbf{n}$ is the tangential velocity (a vector). Moreover, $\gamma > 0$ is a viscous drag coefficient, \mathbf{g} a given surface mechanical load, h is a prescribed inward boundary flux.

To unveil the energetic structure of System (2.1), we derive the structural a priori estimates that we will also use in the analytical part of this paper. To this effect, we report for the reader’s sake some calculations to be used for the inertial term in (2.1a) and the higher-order terms in (2.1c,d), which rely on the representation (2.2) of the material time derivative:

Lemma 2.1 For ϱ constant and for any sufficiently smooth velocity field \mathbf{v} with $\mathbf{v} \cdot \mathbf{n} = 0$ on the boundary and any smooth field \mathbf{A} with $(\mathbf{n} \cdot \nabla) \mathbf{A} = 0$ on the boundary, the following integral identities hold:

$$\frac{d}{dt} \int_{\Omega} \frac{\varrho}{2} |\mathbf{v}|^2 dx = \int_{\Omega} \varrho \left(\frac{D\mathbf{v}}{Dt} + \frac{1}{2} (\operatorname{div} \mathbf{v}) \mathbf{v} \right) \cdot \mathbf{v} dx \quad \text{and} \quad (2.5a)$$

$$\frac{d}{dt} \int_{\Omega} \frac{1}{2} |\nabla \mathbf{A}|^2 dx = \int_{\Omega} \left(\frac{1}{2} |\nabla \mathbf{A}|^2 \mathbf{I} - \nabla \mathbf{A} \boxtimes \nabla \mathbf{A} \right) : \mathbf{e}(\mathbf{v}) - \Delta \mathbf{A} : \frac{D\mathbf{A}}{Dt} dx. \quad (2.5b)$$

Sketch of the proof. The first calculation follows from an application of the divergence theorem and of the requirement that the normal component of \mathbf{v} vanishes on the boundary, taking into account that the density ϱ is constant:

$$\int_{\Omega} \varrho \frac{D\mathbf{v}}{Dt} \cdot \mathbf{v} + \frac{\varrho}{2} (\operatorname{div} \mathbf{v}) |\mathbf{v}|^2 dx = \int_{\Omega} \varrho \frac{\partial \mathbf{v}}{\partial t} \cdot \mathbf{v} + \underbrace{\varrho (\mathbf{v} \cdot \nabla) \mathbf{v} \cdot \mathbf{v}}_{=\mathbf{v} \cdot \nabla (\varrho |\mathbf{v}|^2) / 2} + \operatorname{div} \mathbf{v} \frac{\varrho |\mathbf{v}|^2}{2} dx = \frac{d}{dt} \int_{\Omega} \frac{\varrho}{2} |\mathbf{v}|^2 dx + \int_{\Gamma} \frac{\varrho |\mathbf{v}|^2}{2} \underbrace{\mathbf{v} \cdot \mathbf{n}}_{=0} dS.$$

The second identity (2.5b) results from the calculus:

$$\begin{aligned} - \int_{\Omega} \Delta \mathbf{A} : \frac{D\mathbf{A}}{Dt} dx &= \int_{\Omega} \nabla \mathbf{A} : \nabla \frac{D\mathbf{A}}{Dt} dx - \int_{\Gamma} \underbrace{(\mathbf{n} \cdot \nabla) \mathbf{A}}_{=0} : \frac{D\mathbf{A}}{Dt} dS = \int_{\Omega} \nabla \mathbf{A} : \nabla \frac{\partial \mathbf{A}}{\partial t} + \nabla \mathbf{A} : \nabla (\mathbf{v} \cdot \nabla) \mathbf{A} dx \\ &= \frac{d}{dt} \int_{\Omega} \frac{1}{2} |\nabla \mathbf{A}|^2 dx + \int_{\Omega} \nabla \mathbf{A} \boxtimes \nabla \mathbf{A} : \nabla \mathbf{v} + \underbrace{\nabla \mathbf{A} : (\mathbf{v} \cdot \nabla) \nabla \mathbf{A}}_{=\mathbf{v} \cdot \nabla |\nabla \mathbf{A}|^2 / 2} dx \\ &= \frac{d}{dt} \int_{\Omega} \frac{1}{2} |\nabla \mathbf{A}|^2 dx + \int_{\Omega} \left(\nabla \mathbf{A} \boxtimes \nabla \mathbf{A} - \frac{1}{2} |\nabla \mathbf{A}|^2 \mathbf{I} \right) : \mathbf{e}(\mathbf{v}) dx + \int_{\Gamma} \frac{1}{2} |\nabla \mathbf{A}|^2 \underbrace{\mathbf{v} \cdot \mathbf{n}}_{=0} dS. \quad \square \end{aligned}$$

Let us remark that the above result (2.5a) is consistent with the interpretation proposed in [40] of Temam's extra force $\varrho (\operatorname{div} \mathbf{v}) \mathbf{v} / 2$ [38, 39], an interpretation based on the requirement that the power expenditure of the inertial force be equal to minus the rate of change of kinetic energy [27].

Taking now the scalar product of both sides of (2.1a) with the velocity field \mathbf{v} and integrating over Ω and using standard divergence identities, as well as (2.5a), in combination with the boundary condition (2.4a) and (2.4b) we obtain:

$$0 = \frac{d}{dt} \int_{\Omega} \frac{\varrho}{2} |\mathbf{v}|^2 dx + \int_{\Omega} k_v \mathbf{e}(\mathbf{v}) : \mathbf{e}(\mathbf{v}) + (\mathbf{S} + \mathbf{S}_{\text{str}}) : \mathbf{e}(\mathbf{v}) - \mathbf{f} \cdot \mathbf{v} dx + \int_{\Gamma} \gamma |\mathbf{v}|^2 - \mathbf{g} \cdot \mathbf{v} dS. \quad (2.6)$$

On testing (2.1c) by $\frac{D\mathbf{\Pi}}{Dt}$ and using, in the order, the boundary conditions for $\mathbf{S} = \partial_{\mathbf{E}}(\mathbf{E}, \alpha, \chi)$, $\mathbf{\Pi}$, and \mathbf{v} imposed with (2.4c) and (2.4a), respectively, making also use of (2.5b) for $\mathbf{A} = \mathbf{\Pi}$, we obtain:

$$\begin{aligned} 0 &= \frac{d}{dt} \int_{\Omega} \frac{k_p}{2} |\nabla \mathbf{\Pi}|^2 dx + \int_{\Omega} \partial_{\mathbf{E}} \varphi(\mathbf{E}, \alpha, \chi) : \frac{D\mathbf{E}}{Dt} + \partial_{\frac{D\alpha}{Dt}} \zeta \left(\alpha, \chi; \frac{D\mathbf{\Pi}}{Dt}, \frac{D\alpha}{Dt} \right) : \frac{D\mathbf{\Pi}}{Dt} + k_e \nabla \mathbf{S} : \nabla \mathbf{S} \\ &\quad + \left(k_p \nabla \mathbf{\Pi} \boxtimes \nabla \mathbf{\Pi} - \frac{k_p}{2} |\nabla \mathbf{\Pi}|^2 \mathbf{I} \right) : \mathbf{e}(\mathbf{v}) - \mathbf{S} : \mathbf{e}(\mathbf{v}) dx. \quad (2.7) \end{aligned}$$

In a similar fashion, we operate on (2.1d), using $\frac{D\alpha}{Dt}$ as test function and (2.5b) for $\mathbf{A} = \alpha$, and taking into account the Neumann boundary condition for α in (2.4c). The result is:

$$\begin{aligned} 0 &= \frac{d}{dt} \int_{\Omega} \frac{k_a}{2} |\nabla \alpha|^2 dx + \int_{\Omega} \partial_{\alpha} \varphi(\mathbf{E}, \alpha, \chi) : \frac{D\alpha}{Dt} + \partial_{\frac{D\alpha}{Dt}} \zeta \left(\alpha, \chi; \frac{D\mathbf{\Pi}}{Dt}, \frac{D\alpha}{Dt} \right) \cdot \frac{D\alpha}{Dt} \\ &\quad + \left(k_a \nabla \alpha \boxtimes \nabla \alpha - \frac{k_a}{2} |\nabla \alpha|^2 \mathbf{I} \right) : \mathbf{e}(\mathbf{v}) dx. \quad (2.8) \end{aligned}$$

Finally, on testing the equations in (2.1e), respectively, by μ and $\frac{D\chi}{Dt}$, and adding the resulting equations, we obtain

$$0 = \int_{\Omega} \partial_{\chi} \varphi(\mathbf{E}, \alpha, \chi) \frac{D\chi}{Dt} + \mathbb{M} \nabla \mu : \nabla \mu dx - \int_{\Gamma} h \mu dS, \quad (2.9)$$

where we used also the boundary condition (2.4d). By summing the estimates (2.7)–(2.9) and by observing that

$$\begin{aligned}
& \int_{\Omega} \partial_{\mathbf{E}}\varphi(\mathbf{E}, \alpha, \chi) : \frac{D\mathbf{E}}{Dt} + \partial_{\alpha}\varphi(\mathbf{E}, \alpha, \chi) : \frac{D\alpha}{Dt} + \partial_{\chi}\varphi(\mathbf{E}, \alpha, \chi) : \frac{D\chi}{Dt} dx \\
&= \int_{\Omega} \frac{\partial\varphi_{\mathbf{E}}(\mathbf{E}, \alpha, \chi)}{\partial t} + (\mathbf{v} \cdot \nabla(\varphi_{\mathbf{E}}(\mathbf{E}, \alpha, \chi))) dx \\
&= \frac{d}{dt} \int_{\Omega} \frac{\partial\varphi_{\mathbf{E}}(\mathbf{E}, \alpha, \chi)}{\partial t} dx - \int_{\Omega} \varphi_{\mathbf{E}}(\mathbf{E}, \alpha, \chi) \mathbf{I} : \mathbf{e}(\mathbf{v}) dx + \int_{\Gamma} \varphi_{\mathbf{E}}(\mathbf{E}, \alpha, \chi) \underbrace{\mathbf{v} \cdot \mathbf{n}}_{=0} dS, \quad (2.10)
\end{aligned}$$

we obtain the partial energy balance

$$\begin{aligned}
& \frac{d}{dt} \int_{\Omega} \varphi(\mathbf{E}, \alpha, \chi) + \frac{k_p}{2} |\nabla \mathbf{II}|^2 + \frac{k_a}{2} |\nabla \alpha|^2 dx \\
&+ \int_{\Omega} \frac{\partial_{\frac{D\mathbf{II}}{Dt}}}{\partial t} \zeta \left(\alpha, \chi; \frac{D\mathbf{II}}{Dt}, \frac{D\alpha}{Dt} \right) : \frac{D\mathbf{II}}{Dt} + \partial_{\alpha} \zeta \left(\alpha, \chi; \frac{D\mathbf{II}}{Dt}, \frac{D\alpha}{Dt} \right) : \frac{D\alpha}{Dt} + k_e \nabla \mathbf{S} : \nabla \mathbf{S} - (\mathbf{S} + \mathbf{S}_{\text{str}}) : \mathbf{e}(\mathbf{v}) dx \\
&= \int_{\Gamma} h\mu dS. \quad (2.11)
\end{aligned}$$

On summing (2.11) and (2.6), the contributions from the thermodynamic stress \mathbf{S} and the structural stress \mathbf{S}_{str} cancel, and we arrive at the following total energy balance:

$$\begin{aligned}
& \int_{\Omega} \underbrace{\frac{\rho}{2} |\mathbf{v}(t)|^2}_{\text{kinetic energy}} + \underbrace{\varphi(\mathbf{E}(t), \alpha(t), \chi(t)) + \frac{k_p}{2} |\nabla \mathbf{II}|^2 + \frac{k_a}{2} |\nabla \alpha(t)|^2}_{\text{stored energy}} dx \\
&+ \int_0^t \left(\int_{\Omega} \underbrace{\xi \left(\alpha, \chi; \frac{D\mathbf{II}}{Dt}, \frac{D\alpha}{Dt} \right) + k_v |\mathbf{e}(\mathbf{v})|^2 + \mathbb{M}(\alpha, \chi) \nabla \mu \cdot \nabla \mu + k_e |\nabla \mathbf{S}|^2}_{\text{bulk dissipation rate}} dx + \int_{\Gamma} \underbrace{\gamma |\mathbf{v}_t|^2}_{\text{boundary dissipation rate}} dS \right) dt \\
&= \int_0^t \left(\int_{\Omega} \underbrace{\mathbf{f} \cdot \mathbf{v}}_{\text{power of bulk load}} dx + \int_{\Gamma} \underbrace{\mathbf{g}_t \cdot \mathbf{v}_t + h\mu}_{\text{power of boundary load}} dS \right) dt \\
&+ \int_{\Omega} \frac{\rho}{2} |\mathbf{v}_0|^2 + \varphi(\mathbf{E}_0, \alpha_0, \chi_0) + \frac{k_p}{2} |\nabla \mathbf{II}|^2 + \frac{k_a}{2} |\nabla \alpha_0|^2 dx, \quad (2.12)
\end{aligned}$$

where we abbreviated

$$\xi(\alpha, \chi; \frac{D\mathbf{II}}{Dt}, \frac{D\alpha}{Dt}) = \frac{\partial_{\frac{D\mathbf{II}}{Dt}}}{\partial t} \zeta \left(\alpha, \chi; \frac{D\mathbf{II}}{Dt}, \frac{D\alpha}{Dt} \right) : \frac{D\mathbf{II}}{Dt} + \frac{\partial_{\frac{D\alpha}{Dt}}}{\partial t} \zeta \left(\alpha, \chi; \frac{D\mathbf{II}}{Dt}, \frac{D\alpha}{Dt} \right) \cdot \frac{D\alpha}{Dt}. \quad (2.13)$$

Remark 2.2 (Additive elasto-inelastic strain rate decomposition) The concept of additive strain-rate of the type like (2.1b) dates back basically to Hill [13] and Prager [28]. This concept has been widely used in literature, cf. e.g. [11, Sec.8.6] or [25, Sec.8.3]. For discussing some limitations, see e.g. [14]. It should however be noticed that our equation (2.1b) only mimicks the standard additive decomposition in a simplified way that we write such decomposition like in the small strain setting.

Remark 2.3 (Objective variant.) There is another simplification we used in the decomposition (2.1b). It is known that the material derivatives of tensors \mathbf{E} and \mathbf{II} used in (2.1b-d) are not frame indifferent, and that one should rather use Oldroyd's or Zaremba-Jaumann's or Green-Naghdi's time derivative, cf. e.g. [12, 11, 24]. See also [37, Sect. 5.4]. Nevertheless, the simplifying purely convective but non-objective variant used here is also often exploited in geophysical modelling, although the corresponding symmetric structural stress is usually not reflected correctly there, cf. e.g. [3, 22, 29]. A certain legitimacy of this simplification is in stratified simple-shear situations, as articulated in [31, Proposition 1]. There it is shown, starting from the Kröner-Lee multiplicative decomposition [16, 18], that if displacements are large in one direction (parallel to the stratification) and if the plastic part is a simple shear, then the additive decomposition holds up to higher-order terms. Cf. also [12, Sect. 8.1.3] or [42, Sect. 5]. Actually, the objective variant (most suitably using Zaremba-Jaumann derivatives) is amenable for analysis, too; cf. [34, 35].

Remark 2.4 (The stress diffusion in (2.1b).) The $k_e \Delta \partial_{\mathbf{E}} \varphi$ -term in the “geometrical” equation might be rather controversial. In fluid dynamics, this stress diffusion was advocated in series of works by H. Brenner, cf. e.g. [5,6]. Independently, such diffusion was used also in [2,19]. Cf. also a discussion in [26] and a thermodynamical justification in [41]. It is also somehow similar to the recent proposal in [7]. When omitting information about $\nabla \partial_{\mathbf{E}} \varphi$ and thus about $\nabla \mathbf{E}$, our a priori estimates would work even with $k_e = 0$. A positive (even arbitrarily small) $k_e > 0$ together with the convexity of $\varphi(\cdot, \alpha, \chi)$ is needed only for facilitating convergence of approximate solutions. Therefore, this questionable regularizing diffusion will not essentially influence global energetics and presumably would not be seen on numerically stable algorithms if $k_e > 0$ would be small.

3 Analysis of an initial-value problem for (2.1)–(2.4)

We carry out the analysis for $d \leq 3$. We will now consider the evolution governed by the system (2.1) on a fixed time interval $I = [0, T]$ and complete (2.1) by initial conditions

$$\mathbf{v}|_{t=0} = \mathbf{v}_0, \quad \mathbf{E}|_{t=0} = \mathbf{E}_0, \quad \mathbf{\Pi}|_{t=0} = \mathbf{\Pi}_0, \quad \alpha|_{t=0} = \alpha_0, \quad \chi|_{t=0} = \chi_0. \quad (3.1)$$

We will perform the analysis by a rather constructive approximation, which can in principle be also used as a conceptual numerical algorithm for which we will prove numerical stability and convergence at least in terms of subsequences of approximate solutions.

We will use the standard notation concerning the Lebesgue and the Sobolev spaces, namely $L^p(\Omega; \mathbb{R}^n)$ for Lebesgue measurable functions $\Omega \rightarrow \mathbb{R}^n$ whose Euclidean norm is integrable with p -power, and $W^{k,p}(\Omega; \mathbb{R}^n)$ for functions from $L^p(\Omega; \mathbb{R}^n)$ whose all derivative up to the order k have their Euclidean norm integrable with p -power. Moreover, for a Banach space X and for $I = [0, T]$, we will use the notation $L^p(I; X)$ for the Bochner space of Bochner measurable functions $I \rightarrow X$ whose norm is in $L^p(I)$, and $W^{1,p}(I; X)$ for functions $I \rightarrow X$ whose distributional derivative is in $L^p(I; X)$. Furthermore, $C_w(I; X)$ will denote the Banach space of weakly continuous functions $I \rightarrow X$. We also write briefly $H^k = W^{k,2}$.

We will assume, with some $\epsilon > 0$ arbitrarily small, that

$$\varphi : \mathbb{R}^{d \times d} \times \mathbb{R}^\ell \times \mathbb{R} \rightarrow \mathbb{R} \text{ twice continuously differentiable, bounded from below with} \\ |\partial \varphi(\mathbf{E}, \alpha, \chi)| \leq (1 + |\mathbf{E}|^{3/2-\epsilon} + |\alpha|^{3-\epsilon} + |\chi|^{3-\epsilon})/\epsilon \quad \text{and} \quad (3.2a)$$

$$(\mathbf{E}, \alpha, \chi) \mapsto \varphi(\mathbf{E}, \alpha, \chi) + \frac{1}{2\epsilon} |\alpha|^2 \text{ is convex,} \quad (3.2b)$$

$$\partial_{\mathbf{E}\alpha}^2 \varphi, \partial_{\mathbf{E}\chi}^2 \varphi \text{ bounded and } \varphi(\cdot, \alpha, \cdot) \text{ uniformly convex, i.e. } \forall \mathbf{E}, \tilde{\mathbf{E}} \in \mathbb{R}_{\text{sym}}^{d \times d}, \alpha \in \mathbb{R}^\ell, \chi, \tilde{\chi} \in \mathbb{R} :$$

$$\partial_{\mathbf{E}\mathbf{E}}^2 \varphi(\mathbf{E}, \alpha, \chi) \tilde{\mathbf{E}} : \tilde{\mathbf{E}} + 2\partial_{\mathbf{E}\chi}^2 \varphi(\mathbf{E}, \alpha, \chi) \tilde{\mathbf{E}} \tilde{\chi} + \partial_{\chi\chi}^2 \varphi(\mathbf{E}, \alpha, \chi) \tilde{\chi}^2 \geq \epsilon (|\tilde{\mathbf{E}}|^2 + |\tilde{\chi}|^2), \quad (3.2c)$$

$$\zeta : \mathbb{R}^\ell \times \mathbb{R} \times \mathbb{R}^{d \times d} \times \mathbb{R}^\ell \rightarrow \mathbb{R} \text{ continuous with}$$

$$\zeta(\alpha, \chi; \cdot, \cdot) : \mathbb{R}^{d \times d} \times \mathbb{R}^\ell \rightarrow \mathbb{R} \text{ convex and}$$

$$\zeta(\alpha, \chi; \cdot, \dot{\alpha}) : \mathbb{R}^{d \times d} \setminus \{0\} \rightarrow \mathbb{R} \text{ is continuously differentiable,} \quad (3.2d)$$

$$\zeta(\alpha, \chi; \dot{\mathbf{H}}, \cdot) : \mathbb{R}^\ell \setminus \{0\} \rightarrow \mathbb{R} \text{ is continuously differentiable,} \quad (3.2e)$$

$$\epsilon (|\dot{\mathbf{H}}|^2 + |\dot{\alpha}|^2) \leq \zeta(\alpha, \chi; \dot{\mathbf{H}}, \dot{\alpha}) \leq (1 + |\dot{\mathbf{H}}|^2 + |\dot{\alpha}|^2)/\epsilon, \quad (3.2f)$$

$$\mathbb{M} : \mathbb{R}^\ell \times \mathbb{R} \rightarrow \mathbb{R}^{d \times d} \text{ continuous, bounded, uniformly positive definite,} \quad (3.2g)$$

$$\rho, k_v, k_p, k_a, k_e > 0, \quad (3.2h)$$

$$\mathbf{v}_0 \in L^2(\Omega; \mathbb{R}^d), \quad \mathbf{\Pi}_0 \in H^1(\Omega; \mathbb{R}_{\text{sym}}^{d \times d}), \quad \alpha_0 \in H^1(\Omega; \mathbb{R}^\ell), \quad \chi_0 \in L^2(\Omega), \quad (3.2i)$$

$$\mathbf{f} \in L^1(I; L^2(\Omega; \mathbb{R}^d)), \quad \mathbf{g} \in L^2(I; L^\infty(I; \mathbb{R}^d)), \quad h \in L^2(I; L^{6/5}(I)). \quad (3.2j)$$

We have formulated our growth assumption (3.2c) to be valid for $d = 3$ and $d = 2$ too, but for the latter case it can be weakened. Let us emphasize that we do not assume φ convex, which allows to treat real damage model where φ is always nonconvex, cf. also Remark 4.1.

Definition 3.1 (Weak solutions to (2.1)–(2.4) with (3.1).) The 5-tuple $(\mathbf{v}, \mathbf{E}, \mathbf{\Pi}, \alpha, \chi)$ with

$$\mathbf{v} \in C_w(I; L^2(\Omega; \mathbb{R}^d)) \cap L^2(I; H^1(\Omega; \mathbb{R}^d)) \quad \text{with } \mathbf{n} \cdot \mathbf{v} = 0 \text{ on } I \times \Gamma, \quad (3.3a)$$

$$\mathbf{E} \in C_w(I; L^2(\Omega; \mathbb{R}^{d \times d}_{\text{sym}})) \cap L^2(I; H^1(\Omega; \mathbb{R}^{d \times d}_{\text{sym}})), \quad (3.3b)$$

$$\mathbf{\Pi} \in C_w(I; H^1(\Omega; \mathbb{R}^{d \times d}_{\text{sym}})) \cap W^{1,4/3}(I; L^2(\Omega; \mathbb{R}^{d \times d}_{\text{sym}})), \quad (3.3c)$$

$$\alpha \in C_w(I; H^1(\Omega; \mathbb{R}^\ell)) \cap W^{1,4/3}(I; L^2(\Omega; \mathbb{R}^\ell)), \quad (3.3d)$$

$$\chi \in C_w(I; L^2(\Omega)), \quad \text{and } \mu = \partial_\chi(\mathbf{E}, \alpha, \chi) \in L^2(I; H^1(\Omega)) \quad (3.3e)$$

will be called a weak solution to the boundary-value problem (2.1)–(2.4) with the initial conditions (3.1) if $\mathbf{S} = \partial_{\mathbf{E}}\varphi(\mathbf{E}, \alpha, \chi) \in L^2(I; H^1(\Omega; \mathbb{R}^{d \times d}))$, $\partial_\alpha\varphi(\mathbf{E}, \alpha, \chi) \in L^2(I \times \Omega; \mathbb{R}^\ell)$, and the following four integral identities hold:

$$\begin{aligned} \int_0^T \int_\Omega \varrho \left((\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{1}{2} (\operatorname{div} \mathbf{v}) \mathbf{v} \right) \cdot \tilde{\mathbf{v}} + (\mathbf{S} + \mathbf{S}_{\text{str}}) : \mathbf{e}(\tilde{\mathbf{v}}) + k_v \mathbf{e}(\mathbf{v}) : \mathbf{e}(\tilde{\mathbf{v}}) - \varrho \mathbf{v} \cdot \frac{\partial \tilde{\mathbf{v}}}{\partial t} \, dx dt \\ = \int_\Omega \varrho \mathbf{v}_0 \cdot \tilde{\mathbf{v}}(0) \, dx + \int_0^T \int_\Omega \mathbf{f} \cdot \tilde{\mathbf{v}} \, dx dt + \int_0^T \int_\Gamma \mathbf{g}_t \cdot \tilde{\mathbf{v}}_t \, dS dt \end{aligned} \quad (3.4a)$$

with \mathbf{S}_{str} from (2.3) for all $\tilde{\mathbf{v}} \in H^1(I \times \Omega; \mathbb{R}^d)$ with $\mathbf{n} \cdot \tilde{\mathbf{v}} = 0$ on $I \times \Gamma$ and with $\tilde{\mathbf{v}}(T) = 0$,

$$\int_0^T \int_\Omega (\mathbf{v} \cdot \nabla) (\mathbf{E} + \mathbf{\Pi}) : \tilde{\mathbf{E}} + k_e \nabla \mathbf{S} : \nabla \tilde{\mathbf{E}} - (\mathbf{E} + \mathbf{\Pi}) : \frac{\partial \tilde{\mathbf{E}}}{\partial t} \, dx dt = \int_\Omega (\mathbf{E}_0 + \mathbf{\Pi}_0) : \tilde{\mathbf{E}}(0) \, dx \quad (3.4b)$$

holds for all $\tilde{\mathbf{E}} \in H^1(I \times \Omega; \mathbb{R}^{d \times d})$ with $\tilde{\mathbf{E}}(T, \cdot) = 0$,

$$\begin{aligned} \int_0^T \int_\Omega \zeta(\alpha, \chi; \tilde{\mathbf{\Pi}}, \tilde{\alpha}) - \mathbf{S} : \left(\tilde{\mathbf{\Pi}} - \frac{D\tilde{\mathbf{\Pi}}}{Dt} \right) + \partial_\alpha\varphi(\mathbf{E}, \alpha, \chi) \cdot \left(\tilde{\alpha} - \frac{D\tilde{\alpha}}{Dt} \right) + k_p \nabla \mathbf{\Pi} : \nabla \tilde{\mathbf{\Pi}} \\ + k_p \Delta \mathbf{\Pi} : (\mathbf{v} \cdot \nabla) \mathbf{\Pi} + k_a \nabla \alpha : \nabla \tilde{\alpha} + k_a \Delta \alpha \cdot (\mathbf{v} \cdot \nabla) \alpha \, dx dt + \int_\Omega \frac{k_p}{2} |\nabla \mathbf{\Pi}_0|^2 + \frac{k_a}{2} |\nabla \alpha_0|^2 \, dx \\ \geq \int_\Omega \frac{k_p}{2} |\nabla \mathbf{\Pi}(T)|^2 + \frac{k_a}{2} |\nabla \alpha(T)|^2 \, dx + \int_0^T \int_\Omega \zeta\left(\alpha, \chi; \frac{D\mathbf{\Pi}}{Dt}, \frac{D\alpha}{Dt}\right) \, dx dt \end{aligned} \quad (3.4c)$$

holds for all $(\tilde{\mathbf{\Pi}}, \tilde{\alpha}) \in L^2(I; H^1(\Omega; \mathbb{R}^{d \times d} \times \mathbb{R}^\ell))$, and

$$\int_0^T \int_\Omega (\mathbb{M}(\alpha, \chi) \nabla \mu - \chi \mathbf{v}) \cdot \nabla z - \chi \frac{\partial z}{\partial t} - (\operatorname{div} \mathbf{v}) \chi z \, dx = \int_0^T \int_\Gamma h z \, dS dt + \int_\Omega \chi_0 z \, dx \quad (3.4d)$$

holds for all $z \in C^1(I \times \Omega)$ with $z|_{t=T} = 0$ and with $\mu = \partial_\chi\varphi(\mathbf{E}, \alpha, \chi)$ a.e. in $I \times \Omega$.

Let us note that, for the inequality (3.4c), we used the standard definition of the convex subdifferential of $\zeta(\alpha, \chi; \cdot, \cdot)$ combined with the calculus

$$\int_0^T \int_\Omega \Delta \mathbf{\Pi} : \left(\tilde{\mathbf{\Pi}} - \frac{D\tilde{\mathbf{\Pi}}}{Dt} \right) \, dx dt = \frac{1}{2} \int_\Omega |\nabla \mathbf{\Pi}(T)|^2 - |\nabla \mathbf{\Pi}(0)|^2 \, dx - \int_0^T \int_\Omega \nabla \mathbf{\Pi} : \nabla \tilde{\mathbf{\Pi}} + \Delta \mathbf{\Pi} : (\mathbf{v} \cdot \nabla) \mathbf{\Pi} \, dx dt;$$

for the analytical legitimacy of this formula if $\Delta \mathbf{\Pi}$ and $\frac{D\mathbf{\Pi}}{Dt}$ belong to $L^2(I \times \Omega; \mathbb{R}^{d \times d})$ see e.g. [30, Formula (12.133b)]. An analogous calculus for α , which both will be actually legitimate when showing that both $\Delta \mathbf{\Pi}$ and $\frac{D\mathbf{\Pi}}{Dt}$ belong to $L^2(I \times \Omega; \mathbb{R}^{d \times d})$ and similarly both $\Delta \alpha$ and $\frac{D\alpha}{Dt}$ belong to $L^2(I \times \Omega; \mathbb{R}^\ell)$. The inequality in (3.4c) is also well consistent with the weak continuity in (3.3c,d) and thus weak lower semicontinuity of the right-hand side of (3.4c).

Theorem 3.2 (Existence of weak solutions) *Let $\Omega \subset \mathbb{R}^d$ be Lipschitz and the assumptions (3.2) hold. Then there exists at least one weak solution $(\mathbf{v}, \mathbf{E}, \mathbf{\Pi}, \alpha, \chi)$ to the initial-boundary-value problem (2.1)–(2.4) with (3.1) according the Definition 3.1 which, moreover, satisfies also $\Delta \mathbf{\Pi} \in L^2(I \times \Omega; \mathbb{R}^{d \times d})$, $\Delta \alpha \in L^2(I \times \Omega; \mathbb{R}^\ell)$, and $\nabla \chi \in L^2(I \times \Omega; \mathbb{R}^d)$.*

Sketch of the proof. For clarity, we divide the proof into four steps.

Step 1. (Approximate solutions - existence): We use the Rothe method, i.e. the fully implicit time discretisation with an equidistant partition of the time interval I with the time step $\tau > 0$. We denote by \mathbf{v}_τ^k , \mathbf{E}_τ^k , ... the approximate values of \mathbf{v} , \mathbf{E} , ... at time $k\tau$ with $k = 1, 2, \dots, T/\tau$. We use the notation for the discretised convective time derivative

$$\frac{D_{k-1}^k(\cdot)}{D_\tau t} := \frac{(\cdot)_\tau^k - (\cdot)_\tau^{k-1}}{\tau} + (\mathbf{v}_\tau^k \cdot \nabla)(\cdot)_\tau^k,$$

i.e. e.g. $\frac{D_{k-1}^k \mathbf{v}}{D_\tau t}$ will mean $\frac{\mathbf{v}_\tau^k - \mathbf{v}_\tau^{k-1}}{\tau} + (\mathbf{v}_\tau^k \cdot \nabla) \mathbf{v}_\tau^k$ etc. With this notation, we consider the scheme

$$\varrho \frac{D_{k-1}^k \mathbf{v}}{D_\tau t} = \operatorname{div}(\mathbf{S}_\tau^k + \mathbf{S}_{\text{str},\tau}^k + k_v \mathbf{e}(\mathbf{v}_\tau^k)) + \mathbf{f}_\tau^k - \frac{\varrho}{2} (\operatorname{div} \mathbf{v}_\tau^k) \mathbf{v}_\tau^k, \quad (3.5a)$$

$$\frac{D_{k-1}^k \mathbf{E}}{D_\tau t} = \mathbf{e}(\mathbf{v}_\tau^k) - \frac{D_{k-1}^k \mathbf{\Pi}}{D_\tau t} + k_e \Delta \mathbf{S}_\tau^k \quad \text{with} \quad \mathbf{S}_\tau^k = \partial_{\mathbf{E}} \varphi(\mathbf{E}_\tau^k, \alpha_\tau^k, \chi_\tau^k), \quad (3.5b)$$

$$\partial_{\frac{D_{k-1}^k \mathbf{\Pi}}{D_\tau t}} \zeta \left(\alpha_\tau^{k-1}, \chi_\tau^{k-1}, \frac{D_{k-1}^k \mathbf{\Pi}}{D_\tau t}, \frac{D_{k-1}^k \alpha}{D_\tau t} \right) - \mathbf{S}_\tau^k \ni k_p \Delta \mathbf{\Pi}_\tau^k, \quad (3.5c)$$

$$\partial_{\frac{D_{k-1}^k \alpha}{D_\tau t}} \zeta \left(\alpha_\tau^{k-1}, \chi_\tau^{k-1}, \frac{D_{k-1}^k \mathbf{\Pi}}{D_\tau t}, \frac{D_{k-1}^k \alpha}{D_\tau t} \right) + \frac{\alpha_\tau^k - \alpha_\tau^{k-1}}{\sqrt{\tau}} + \partial_\alpha \varphi(\mathbf{E}_\tau^k, \alpha_\tau^k, \chi_\tau^k) \ni k_a \Delta \alpha_\tau^k, \quad (3.5d)$$

$$\frac{D_{k-1}^k \chi}{D_\tau t} = \operatorname{div}(\mathbb{M}(\alpha_\tau^{k-1}, \chi_\tau^{k-1}) \nabla \mu_\tau^k) \quad \text{with} \quad \mu_\tau^k = \partial_\chi \varphi(\mathbf{E}_\tau^k, \alpha_\tau^k, \chi_\tau^k), \quad (3.5e)$$

and with the discrete structural stress

$$\mathbf{S}_{\text{str},\tau}^k = k_p \nabla \mathbf{\Pi}_\tau^k \boxtimes \nabla \mathbf{\Pi}_\tau^k + k_a \nabla \alpha_\tau^k \boxtimes \nabla \alpha_\tau^k - \left(\varphi(\mathbf{E}_\tau^k, \alpha_\tau^k, \chi_\tau^k) + \frac{k_p}{2} |\nabla \mathbf{\Pi}_\tau^k|^2 + \frac{k_a}{2} |\nabla \alpha_\tau^k|^2 \right) \mathbf{I}. \quad (3.5f)$$

The boundary conditions (2.4) are discretised correspondingly, i.e.

$$\mathbf{v}_\tau^k \cdot \mathbf{n} = 0, \quad (3.6a)$$

$$((\mathbf{S}_\tau^k + k_v \mathbf{e}(\mathbf{v}_\tau^k) + \mathbf{S}_{\text{str},\tau}^k) \mathbf{n})_{\mathfrak{t}} + \gamma (\mathbf{v}_\tau^k)_{\mathfrak{t}} = (\mathbf{g}_\tau^k)_{\mathfrak{t}}, \quad (3.6b)$$

$$(\mathbf{n} \cdot \nabla) \mathbf{S}_\tau^k = 0, \quad (\mathbf{n} \cdot \nabla) \mathbf{\Pi}_\tau^k = 0, \quad \nabla \alpha_\tau^k \cdot \mathbf{n} = 0, \quad (3.6c)$$

$$\mathbb{M}(\alpha_\tau^{k-1}, \chi_\tau^{k-1}) : (\nabla \mu_\tau^k \otimes \mathbf{n}) = h_\tau^k \quad \text{and} \quad \nabla \chi_\tau^k \cdot \mathbf{n} = 0. \quad (3.6d)$$

We used the notation $\mathbf{f}_\tau^k := \int_{(k-1)\tau}^{k\tau} \mathbf{f}(t) dt$ and similarly also for \mathbf{g}_τ^k and h_τ^k . The system of boundary-value problems (3.5)–(3.6) is to be solved recursively for $k = 1, 2, \dots, T/\tau$, assuming T/τ integer, and starting with

$$\mathbf{v}_\tau^0 = \mathbf{v}_0, \quad \mathbf{E}_\tau^0 = \mathbf{E}_0, \quad \mathbf{\Pi}_\tau^0 = \mathbf{\Pi}_0, \quad \alpha_\tau^0 = \alpha_0, \quad \chi_\tau^0 = \chi_0. \quad (3.7)$$

Let us point out that the term $(\alpha_\tau^k - \alpha_\tau^{k-1})/\sqrt{\tau}$ in (3.5d) is devised to convexify φ using (3.2b) for small $\tau > 0$ but it still vanishes in the limit.

For a given $(\mathbf{v}_\tau^{k-1}, \mathbf{E}_\tau^{k-1}, \mathbf{\Pi}_\tau^{k-1}, \alpha_\tau^{k-1}, \chi_\tau^{k-1}) \in H^1(\Omega; \mathbb{R}^d \times \mathbb{R}^{d \times d} \times \mathbb{R}^{d \times d} \times \mathbb{R}^\ell \times \mathbb{R}) =: V$, the existence of weak solutions $(\mathbf{v}_\tau^k, \mathbf{E}_\tau^k, \mathbf{\Pi}_\tau^k, \alpha_\tau^k, \chi_\tau^k) \in V$ of the coupled semi-linear boundary-value problem (3.5)–(3.6) can thus be seen by the application of Galerkin-approximation-based arguments from the theory of coercive weakly continuous set-valued operators from V to Z^* for some $Z \subset V$ with the set-valued part arising from a convex potential; cf. e.g. [30, Sect. 2.5 and 5.3]. Here one should choose $Z = W^{1,\infty}(\Omega; \mathbb{R}^d) \times H^1(\Omega; \mathbb{R}^{d \times d} \times \mathbb{R}^{d \times d} \times \mathbb{R}^\ell \times \mathbb{R})$ to handle the structural stress which belongs to $L^1(\Omega; \mathbb{R}^{d \times d}) \subset W^{1,\infty}(\Omega; \mathbb{R}^{d \times d})^*$ but not to $H^1(\Omega; \mathbb{R}^{d \times d})^*$ in general. (Note that the system does not have any potential because of the convective terms and the related structural stress occurring in (3.5a) make the system nonsymmetric, so that the direct method cannot be used.) The mentioned coercivity is a particular consequence of the a priori estimates derived below. The weak continuity actually makes the components $\mathbf{\Pi}_\tau^k$ and α_τ^k strongly convergent by the arguments like (3.27) below, which is needed for the continuity of the nonlinearly dependent structural stress. Also the classical Relich compact-embedding theorem is used at several places to couple with the lower-order nonlinearities. The L^2 -information about gradients of \mathbf{E}_τ^k and χ_τ^k can be obtained like in (3.24f) below. Moreover, we can rely also on an L^2 -information about $\Delta \mathbf{\Pi}_\tau^k$ and $\Delta \alpha_\tau^k$ like in (3.25f)

below. Thus the equation/inclusions (3.5b,c,d) hold even pointwise a.e. on Ω . We have here additionally $\nabla \mathbf{S}_\tau^k \in L^2(\Omega; \mathbb{R}^{d \times d})$.

Step 2. (Energetics of the discrete solutions): The a priori estimation is based on the energy test. This means here the test of (3.5a) by \mathbf{v}_τ^k while using also (3.5b), then we test the inclusion (3.5c) by $(\mathbf{\Pi}_\tau^k - \mathbf{\Pi}_\tau^{k-1})/\tau + (\mathbf{v}_\tau^k \cdot \nabla) \mathbf{\Pi}_\tau^k$ and the inclusion (3.5d) by $(\alpha_\tau^k - \alpha_\tau^{k-1})/\tau + (\mathbf{v}_\tau^k \cdot \nabla) \alpha_\tau^k$, and we test the particular equations in (3.5e) by μ_τ^k and $(\chi_\tau^k - \chi_\tau^{k-1})/\tau + \mathbf{v}_\tau^k \cdot \nabla \chi_\tau^k$, respectively.

The mentioned tests thus give the energy balance (2.12) written as an inequality for the time-discrete approximation. More specifically, the terms related to inertia in (3.5a) uses the calculus

$$\begin{aligned} \left(\varrho \frac{\mathbf{v}_\tau^k - \mathbf{v}_\tau^{k-1}}{\tau} + \varrho (\mathbf{v}_\tau^k \cdot \nabla) \mathbf{v}_\tau^k - \mathbf{f}_{\text{str},\tau}^k \right) \cdot \mathbf{v}_\tau^k &= \frac{\varrho}{2} \frac{|\mathbf{v}_\tau^k|^2 - |\mathbf{v}_\tau^{k-1}|^2}{\tau} + \varrho (\mathbf{v}_\tau^k \cdot \nabla) \mathbf{v}_\tau^k \cdot \mathbf{v}_\tau^k \\ &\quad + \frac{\varrho}{2} (\text{div } \mathbf{v}_\tau^k) |\mathbf{v}_\tau^k|^2 + \tau \frac{\varrho}{2} \left| \frac{\mathbf{v}_\tau^k - \mathbf{v}_\tau^{k-1}}{\tau} \right|^2 \end{aligned} \quad (3.8)$$

with the ‘‘structural’’ force $\mathbf{f}_{\text{str},\tau}^k := -\frac{1}{2} \varrho (\text{div } \mathbf{v}_\tau^k) \mathbf{v}_\tau^k$, cf. the last term in (3.5a). This holds pointwise, and, when integrated over Ω , we further use also

$$\int_{\Omega} \varrho (\mathbf{v}_\tau^k \cdot \nabla) \mathbf{v}_\tau^k \cdot \mathbf{v}_\tau^k \, dx = - \int_{\Omega} \frac{\varrho}{2} |\mathbf{v}_\tau^k|^2 (\text{div } \mathbf{v}_\tau^k) \, dx + \int_{\Gamma} \frac{\varrho}{2} |\mathbf{v}_\tau^k|^2 (\mathbf{v}_\tau^k \cdot \mathbf{n}) \, dS. \quad (3.9)$$

The last term in (3.8) is non-negative and will simply be forgotten, which gives a discrete analog of (2.5a) as the inequality

$$\begin{aligned} \int_{\Omega} \left(\varrho \frac{\mathbf{v}_\tau^k - \mathbf{v}_\tau^{k-1}}{\tau} + \varrho (\mathbf{v}_\tau^k \cdot \nabla) \mathbf{v}_\tau^k - \mathbf{f}_{\text{str},k} \right) \cdot \mathbf{v}_\tau^k \, dx &\stackrel{(3.8)}{\geq} \int_{\Omega} \left(\frac{\varrho}{2} \frac{|\mathbf{v}_\tau^k|^2 - |\mathbf{v}_\tau^{k-1}|^2}{\tau} \right. \\ &\quad \left. + \varrho (\mathbf{v}_\tau^k \cdot \nabla) \mathbf{v}_\tau^k \cdot \mathbf{v}_\tau^k + \frac{\varrho}{2} (\text{div } \mathbf{v}_\tau^k) |\mathbf{v}_\tau^k|^2 \right) \, dx \stackrel{(3.9)}{=} \int_{\Omega} \frac{\varrho}{2} \frac{|\mathbf{v}_\tau^k|^2 - |\mathbf{v}_\tau^{k-1}|^2}{\tau} \, dx + \int_{\Gamma} \frac{\varrho}{2} |\mathbf{v}_\tau^k|^2 (\mathbf{v}_\tau^k \cdot \mathbf{n}) \, dS. \end{aligned} \quad (3.10)$$

The last term vanishes due to the boundary condition (3.6a). The further term in (3.5a) uses the calculus

$$\begin{aligned} \int_{\Omega} \text{div } \mathbf{S}_\tau^k \cdot \mathbf{v}_\tau^k \, dx &= \int_{\Gamma} \mathbf{S}_\tau^k : (\mathbf{v}_\tau^k \otimes \mathbf{n}) \, dS - \int_{\Omega} \mathbf{S}_\tau^k : \mathbf{e}(\mathbf{v}_\tau^k) \, dx \\ &\stackrel{(3.5b)}{=} \int_{\Gamma} \mathbf{S}_\tau^k : (\mathbf{v}_\tau^k \otimes \mathbf{n}) \, dS - \int_{\Omega} \mathbf{S}_\tau^k : \frac{D_{k-1}^k \mathbf{E}}{D_\tau t} - \mathbf{S}_\tau^k : \frac{D_{k-1}^k \mathbf{\Pi}}{D_\tau t} - k_e \mathbf{S}_\tau^k : \Delta \mathbf{S}_\tau^k \, dx \\ &= \int_{\Gamma} \mathbf{S}_\tau^k : (\mathbf{v}_\tau^k \otimes \mathbf{n}) + k_e \mathbf{S}_\tau^k : (\mathbf{n} \cdot \nabla) \mathbf{S}_\tau^k \, dS \\ &\quad - \int_{\Omega} \mathbf{S}_\tau^k : \frac{\mathbf{E}_\tau^k - \mathbf{E}_\tau^{k-1}}{\tau} + \mathbf{S}_\tau^k : (\mathbf{v}_\tau^k \cdot \nabla) \mathbf{E}_\tau^k - \mathbf{S}_\tau^k : \frac{D_{k-1}^k \mathbf{\Pi}}{D_\tau t} + k_e |\nabla \mathbf{S}_\tau^k|^2 \, dx, \end{aligned} \quad (3.11)$$

where we abbreviated $\mathbf{S}_\tau^k = \partial_{\mathbf{E}} \varphi(\mathbf{E}_\tau^k, \alpha_\tau^k, \chi_\tau^k)$. Finally, we have

$$\begin{aligned} \int_{\Omega} (\text{div}(\mathbf{S}_{\text{str},\tau}^k + k_v \mathbf{e}(\mathbf{v}_\tau^k)) \cdot \mathbf{v}_\tau^k \, dx \\ = \int_{\Gamma} (\mathbf{S}_{\text{str},\tau}^k + k_v \mathbf{e}(\mathbf{v}_\tau^k)) : (\mathbf{v}_\tau^k \otimes \mathbf{n}) \, dS - \int_{\Omega} (\mathbf{S}_{\text{str},\tau}^k + k_v \mathbf{e}(\mathbf{v}_\tau^k)) : \nabla \mathbf{v}_\tau^k \, dx. \end{aligned} \quad (3.12)$$

The mentioned test (3.5c) by $\frac{D_{k-1}^k \mathbf{\Pi}}{D_\tau t} = \frac{\mathbf{\Pi}_\tau^k - \mathbf{\Pi}_\tau^{k-1}}{\tau} + (\mathbf{v}_\tau^k \cdot \nabla) \mathbf{\Pi}_\tau^k$ gives contributions to the dissipation rate and to the stored-energy rate. The dissipation and the gradient terms in (3.5c) yield, using also a discrete version of the calculus behind (2.5b), that

$$\begin{aligned} \int_{\Omega} \partial_{\frac{D \mathbf{\Pi}}{D t}} \zeta \left(\alpha_\tau^{k-1}, \chi_\tau^{k-1}; \frac{D_{k-1}^k \mathbf{\Pi}}{D_\tau t}, \frac{D_{k-1}^k \alpha}{D_\tau t} \right) : \frac{D_{k-1}^k \mathbf{\Pi}}{D_\tau t} - k_p \Delta \mathbf{\Pi}_\tau^k : \frac{D_{k-1}^k \mathbf{\Pi}}{D_\tau t} \, dx \\ \geq \int_{\Omega} \partial_{\frac{D \mathbf{\Pi}}{D t}} \zeta \left(\alpha_\tau^{k-1}, \chi_\tau^{k-1}; \frac{D_{k-1}^k \mathbf{\Pi}}{D_\tau t}, \frac{D_{k-1}^k \alpha}{D_\tau t} \right) : \frac{D_{k-1}^k \mathbf{\Pi}}{D_\tau t} - k_p (\mathbf{v}_\tau^k \cdot \nabla) \mathbf{\Pi}_\tau^k : \Delta \mathbf{\Pi}_\tau^k \, dx \\ \quad + \int_{\Omega} \frac{k_p}{2} \frac{|\nabla \mathbf{\Pi}_\tau^k|^2 - |\nabla \mathbf{\Pi}_\tau^{k-1}|^2}{\tau} \, dx - \int_{\Gamma} k_p \nabla \mathbf{\Pi}_\tau^k : \left(\frac{D_{k-1}^k \mathbf{\Pi}}{D_\tau t} \otimes \mathbf{n} \right) \, dS \\ = \int_{\Omega} \partial_{\frac{D \mathbf{\Pi}}{D t}} \zeta \left(\alpha_\tau^{k-1}, \chi_\tau^{k-1}; \frac{D_{k-1}^k \mathbf{\Pi}}{D_\tau t}, \frac{D_{k-1}^k \alpha}{D_\tau t} \right) : \frac{D_{k-1}^k \mathbf{\Pi}}{D_\tau t} + \left(k_p \nabla \mathbf{\Pi}_\tau^k \boxtimes \nabla \mathbf{\Pi}_\tau^k - \frac{k_p}{2} |\nabla \mathbf{\Pi}_\tau^k|^2 \mathbf{I} \right) : \mathbf{e}(\mathbf{v}_\tau^k) \, dx \\ \quad + \int_{\Omega} \frac{k_p}{2} \frac{|\nabla \mathbf{\Pi}_\tau^k|^2 - |\nabla \mathbf{\Pi}_\tau^{k-1}|^2}{\tau} \, dx, \end{aligned} \quad (3.13)$$

where the term boundary term $k_p \nabla \mathbf{\Pi}_\tau^k : (\frac{D_{k-1}^k \mathbf{\Pi}}{D_\tau t} \otimes \mathbf{n}) = (\mathbf{n} \cdot \nabla) \mathbf{\Pi}_\tau^k : \frac{D_{k-1}^k \mathbf{\Pi}}{D_\tau t}$ vanishes thanks to (3.6c) and where the inequality relies on the convexity of the functional $\mathbf{\Pi} \mapsto \int_\Omega \frac{k_p}{2} |\nabla \mathbf{\Pi}|^2 dx$. The inequality follows from the calculus

$$\begin{aligned} - \int_\Omega \Delta \mathbf{\Pi}_\tau^k : \frac{\mathbf{\Pi}_\tau^k - \mathbf{\Pi}_\tau^{k-1}}{\tau} dx &= \int_\Omega \nabla \mathbf{\Pi}_\tau^k : \nabla \frac{\mathbf{\Pi}_\tau^k - \mathbf{\Pi}_\tau^{k-1}}{\tau} dx \\ &= \int_\Omega \left(\frac{|\nabla \mathbf{\Pi}_\tau^k|^2 - |\nabla \mathbf{\Pi}_\tau^{k-1}|^2}{2\tau} \right. \\ &\quad \left. + \frac{\tau}{2} \left| \frac{\nabla \mathbf{\Pi}_\tau^k - \nabla \mathbf{\Pi}_\tau^{k-1}}{\tau} \right|^2 \right) dx \geq \int_\Omega \frac{|\nabla \mathbf{\Pi}_\tau^k|^2 - |\nabla \mathbf{\Pi}_\tau^{k-1}|^2}{2\tau} dx. \end{aligned}$$

while the meaning of the expression $\partial_{\frac{D\mathbf{\Pi}}{Dt}} \zeta(\alpha, \chi; \dot{\mathbf{\Pi}}, \dot{\alpha}) : \dot{\mathbf{\Pi}}$ is well defined even if $\partial_{\frac{D\mathbf{\Pi}}{Dt}} \zeta(\alpha, \chi; \cdot, \dot{\alpha})$ is multi-valued at $\dot{\mathbf{\Pi}} = 0$, cf. (3.2d).

Moreover, the test of (3.5d) by $\frac{D_{k-1}^k \alpha}{D_\tau t} = \frac{\alpha_\tau^k - \alpha_\tau^{k-1}}{\tau} + \mathbf{v}_\tau^k \cdot \nabla \alpha_\tau^k$ gives rise to the term $\int_\Omega (\mathbf{v}_\tau^k \cdot \nabla \alpha_\tau^k) \Delta \alpha_\tau^k dx$. By proceeding as in (3.13), using the boundary condition $\mathbf{v}_\tau^k \cdot \mathbf{n} = 0$, we obtain

$$\begin{aligned} &\int_\Omega \partial_{\frac{D\alpha}{Dt}} \zeta(\alpha_\tau^{k-1}, \chi_\tau^{k-1}; \frac{D_{k-1}^k \mathbf{\Pi}}{D_\tau t}, \frac{D_{k-1}^k \alpha}{D_\tau t}) \cdot \frac{D_{k-1}^k \alpha}{D_\tau t} + \left(\partial_\alpha \varphi(\mathbf{E}_\tau^k, \alpha_\tau^k, \chi_\tau^k) + \frac{\alpha_\tau^k - \alpha_\tau^{k-1}}{\sqrt{\tau}} - k_a \Delta \alpha_\tau^k \right) \cdot \frac{D_{k-1}^k \alpha}{D_\tau t} dx \\ &\geq \int_\Omega \partial_{\frac{D\alpha}{Dt}} \zeta(\alpha_\tau^{k-1}, \chi_\tau^{k-1}; \frac{D_{k-1}^k \mathbf{\Pi}}{D_\tau t}, \frac{D_{k-1}^k \alpha}{D_\tau t}) \cdot \frac{D_{k-1}^k \alpha}{D_\tau t} + \left(k_a \nabla \alpha_\tau^k \otimes \nabla \alpha_\tau^k - \frac{k_a}{2} |\nabla \alpha_\tau^k|^2 \mathbf{I} \right) : \mathbf{e}(\mathbf{v}_\tau^k) dx \\ &\quad + \int_\Omega \partial_\alpha \varphi(\mathbf{E}_\tau^k, \alpha_\tau^k, \chi_\tau^k) \cdot \frac{\alpha_\tau^k - \alpha_\tau^{k-1}}{\tau} + \frac{k_a}{2} \frac{|\nabla \alpha_\tau^k|^2 - |\nabla \alpha_\tau^{k-1}|^2}{\tau} dx. \end{aligned} \quad (3.14)$$

The inequality in (3.14) arises from the same reasons as in (3.13) using (3.2e).

Eventually, the test of (3.5e) by μ_τ^k gives

$$\begin{aligned} &\int_\Omega \left(\frac{D_{k-1}^k \chi}{D_\tau t} - \operatorname{div}(\mathbb{M}(\alpha_\tau^{k-1}, \chi_\tau^{k-1}) \nabla \mu_\tau^k) \right) \mu_\tau^k dx \\ &= \int_\Omega \left(\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau} + \mathbf{v}_\tau^k \cdot \nabla \chi_\tau^k \right) \mu_\tau^k + \mathbb{M}(\alpha_\tau^{k-1}, \chi_\tau^{k-1}) \nabla \mu_\tau^k \cdot \nabla \mu_\tau^k dx - \int_\Gamma h \mu_\tau^k dS \\ &\geq \int_\Omega \partial_\chi \varphi(\mathbf{E}_\tau^k, \alpha_\tau^k, \chi_\tau^k) \left(\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau} + \mathbf{v}_\tau^k \cdot \nabla \chi_\tau^k \right) + \mathbb{M}(\alpha_\tau^{k-1}, \chi_\tau^{k-1}) \nabla \mu_\tau^k \cdot \nabla \mu_\tau^k dx - \int_\Gamma h \mu_\tau^k dS. \end{aligned} \quad (3.15)$$

Using the semi-convexity of φ , we can estimate the sum of the three terms arising in (3.11), (3.14), and (3.15) together with the convexifying term in (3.5d) as

$$\begin{aligned} S_\tau^k &: \frac{\mathbf{E}_\tau^k - \mathbf{E}_\tau^{k-1}}{\tau} + \left(\partial_\alpha \varphi(\mathbf{E}_\tau^k, \alpha_\tau^k, \chi_\tau^k) + \frac{\alpha_\tau^k - \alpha_\tau^{k-1}}{\sqrt{\tau}} \right) \cdot \frac{\alpha_\tau^k - \alpha_\tau^{k-1}}{\tau} + \partial_\chi \varphi(\mathbf{E}_\tau^k, \alpha_\tau^k, \chi_\tau^k) \frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau} \\ &= \partial_E \varphi(\mathbf{E}_\tau^k, \alpha_\tau^k, \chi_\tau^k) : \frac{\mathbf{E}_\tau^k - \mathbf{E}_\tau^{k-1}}{\tau} + \left(\partial_\alpha \varphi(\mathbf{E}_\tau^k, \alpha_\tau^k, \chi_\tau^k) + \frac{\alpha_\tau^k}{\sqrt{\tau}} \right) \cdot \frac{\alpha_\tau^k - \alpha_\tau^{k-1}}{\tau} \\ &\quad + \partial_\chi \varphi(\mathbf{E}_\tau^k, \alpha_\tau^k, \chi_\tau^k) \frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau} - \frac{\alpha_\tau^{k-1}}{\sqrt{\tau}} \cdot \frac{\alpha_\tau^k - \alpha_\tau^{k-1}}{\tau} \\ &\geq \frac{\varphi(\mathbf{E}_\tau^k, \alpha_\tau^k, \chi_\tau^k) - \varphi(\mathbf{E}_\tau^{k-1}, \alpha_\tau^{k-1}, \chi_\tau^{k-1})}{\tau} + \frac{1}{2\sqrt{\tau}} \frac{|\alpha_\tau^k|^2 - |\alpha_\tau^{k-1}|^2}{\tau} - \frac{\alpha_\tau^{k-1}}{\sqrt{\tau}} \cdot \frac{\alpha_\tau^k - \alpha_\tau^{k-1}}{\tau} \\ &= \frac{\varphi(\mathbf{E}_\tau^k, \alpha_\tau^k, \chi_\tau^k) - \varphi(\mathbf{E}_\tau^{k-1}, \alpha_\tau^{k-1}, \chi_\tau^{k-1})}{\tau} - \frac{\sqrt{\tau}}{2} \left| \frac{\alpha_\tau^k - \alpha_\tau^{k-1}}{\tau} \right|^2, \end{aligned} \quad (3.16)$$

cf. also the calculation in [30, Remark 8.24]. This holds a.e. on Ω and is to be integrated over Ω . For the remaining three convective terms arising from these tests, we use the calculus

$$\begin{aligned}
& \int_{\Omega} \left(\partial_{\mathbf{E}} \varphi(\mathbf{E}_{\tau}^k, \alpha_{\tau}^k, \chi_{\tau}^k) : (\mathbf{v}_{\tau}^k \cdot \nabla) \mathbf{E}_{\tau}^k + \partial_{\alpha} \varphi(\mathbf{E}_{\tau}^k, \alpha_{\tau}^k, \chi_{\tau}^k) \cdot (\mathbf{v}_{\tau}^k \cdot \nabla \alpha_{\tau}^k) \right. \\
& \quad \left. + \partial_{\chi} \varphi(\mathbf{E}_{\tau}^k, \alpha_{\tau}^k, \chi_{\tau}^k) \cdot (\mathbf{v}_{\tau}^k \cdot \nabla \chi_{\tau}^k) \right) dx = \int_{\Omega} \nabla \varphi(\mathbf{E}_{\tau}^k, \alpha_{\tau}^k, \chi_{\tau}^k) \cdot \mathbf{v}_{\tau}^k dx \\
& \quad = \int_{\Gamma} \varphi(\mathbf{E}_{\tau}^k, \alpha_{\tau}^k, \chi_{\tau}^k) \mathbf{v}_{\tau}^k \cdot \mathbf{n} dS - \int_{\Omega} \varphi(\mathbf{E}_{\tau}^k, \alpha_{\tau}^k, \chi_{\tau}^k) \operatorname{div} \mathbf{v}_{\tau}^k dx \\
& \quad = - \int_{\Gamma} \varphi(\mathbf{E}_{\tau}^k, \alpha_{\tau}^k, \chi_{\tau}^k) \mathbf{I} : \mathbf{e}(\mathbf{v}_{\tau}^k) dx, \tag{3.17}
\end{aligned}$$

which cancels with the pressure-type stress contribution $\varphi(\mathbf{E}_{\tau}^k, \alpha_{\tau}^k, \chi_{\tau}^k) \mathbf{I}$.

Eventually, after summation over $k = 1, 2, \dots$, we obtain (2.12) as an upper estimate up to an error term which is small for $\tau > 0$ small, so that it can be used for a priori estimates. More precisely, by the test of the regularizing term $(\alpha_{\tau}^k - \alpha_{\tau}^{k-1})/\sqrt{\tau}$ by $\frac{D_{\tau}^{k-1}}{D_{\tau} t} \alpha$, we obtain still the term

$$\begin{aligned}
& \int_{\Omega} \frac{\alpha_{\tau}^k - \alpha_{\tau}^{k-1}}{\sqrt{\tau}} \cdot (\mathbf{v}_{\tau}^k \cdot \nabla) \alpha_{\tau}^k dx = \sqrt{\tau} \int_{\Omega} \frac{\alpha_{\tau}^k - \alpha_{\tau}^{k-1}}{\tau} \cdot (\mathbf{v}_{\tau}^k \cdot \nabla) \alpha_{\tau}^k dx \\
& \quad = \sqrt{\tau} \int_{\Omega} \frac{\alpha_{\tau}^k - \alpha_{\tau}^{k-1}}{\tau} \cdot \frac{D_{\tau}^{k-1} \alpha}{D_{\tau} t} - \left| \frac{\alpha_{\tau}^k - \alpha_{\tau}^{k-1}}{\tau} \right|^2 dx \\
& \quad \leq \frac{\sqrt{\tau}}{2} \left\| \frac{D_{\tau}^{k-1} \alpha}{D_{\tau} t} \right\|_{L^2(\Omega; \mathbb{R}^{\ell})}^2 - \frac{\sqrt{\tau}}{2} \left\| \frac{\alpha_{\tau}^k - \alpha_{\tau}^{k-1}}{\tau} \right\|_{L^2(\Omega; \mathbb{R}^{\ell})}^2, \tag{3.18}
\end{aligned}$$

which allows for estimation in the next step when relying on the assumption (3.2f) and on the last term in (3.16).

Step 3. (A priori estimates): Using the values $(\mathbf{v}_{\tau}^k)_{k=0}^{T/\tau}$, we define the piecewise constant and the piecewise affine interpolants respectively as

$$\bar{\mathbf{v}}_{\tau}(t) := \mathbf{v}_{\tau}^k, \quad \underline{\mathbf{v}}_{\tau}(t) := \mathbf{v}_{\tau}^{k-1}, \quad \text{and} \quad \mathbf{v}_{\tau}(t) := \left(\frac{t}{\tau} - k + 1 \right) \mathbf{v}_{\tau}^k + \left(k - \frac{t}{\tau} \right) \mathbf{v}_{\tau}^{k-1} \quad \text{for} \quad (k-1)\tau < t \leq k\tau \tag{3.19}$$

for $k = 0, 1, \dots, T/\tau$. Analogously, we define also $\bar{\mathbf{E}}_{\tau}$, $\bar{\mathbf{E}}_{\tau}$, etc. In terms of such interpolants, we can write the discrete recursive system (3.5) ‘‘compactly’’ as

$$\rho \frac{\partial \bar{\mathbf{v}}_{\tau}}{\partial t} + (\bar{\mathbf{v}}_{\tau} \cdot \nabla) \bar{\mathbf{v}}_{\tau} = \operatorname{div} (\bar{\mathbf{S}}_{\tau} + \bar{\mathbf{S}}_{\text{str}, \tau} + k_{\text{v}} \mathbf{e}(\bar{\mathbf{v}}_{\tau})) + \bar{\mathbf{f}}_{\tau} - \frac{\rho}{2} (\operatorname{div} \bar{\mathbf{v}}_{\tau}) \bar{\mathbf{v}}_{\tau}, \tag{3.20a}$$

$$\frac{\partial \bar{\mathbf{E}}_{\tau}}{\partial t} + (\bar{\mathbf{v}}_{\tau} \cdot \nabla) \bar{\mathbf{E}}_{\tau} = \mathbf{e}(\bar{\mathbf{v}}_{\tau}) - \frac{\partial \bar{\boldsymbol{\Pi}}_{\tau}}{\partial t} - (\bar{\mathbf{v}}_{\tau} \cdot \nabla) \bar{\boldsymbol{\Pi}}_{\tau} + k_{\text{e}} \Delta \bar{\mathbf{S}}_{\tau} \quad \text{with} \quad \bar{\mathbf{S}}_{\tau} = \partial_{\mathbf{E}} \varphi(\bar{\mathbf{E}}_{\tau}, \bar{\alpha}_{\tau}, \bar{\chi}_{\tau}), \tag{3.20b}$$

$$\frac{\partial \bar{\boldsymbol{\Pi}}_{\tau}}{\partial t} \zeta(\underline{\alpha}_{\tau}, \underline{\chi}_{\tau}; \frac{\partial \bar{\boldsymbol{\Pi}}_{\tau}}{\partial t} + (\bar{\mathbf{v}}_{\tau} \cdot \nabla) \bar{\boldsymbol{\Pi}}_{\tau}, \frac{\partial \bar{\alpha}_{\tau}}{\partial t} + (\bar{\mathbf{v}}_{\tau} \cdot \nabla) \bar{\alpha}_{\tau}) - \bar{\mathbf{S}}_{\tau} \ni k_{\text{p}} \Delta \bar{\boldsymbol{\Pi}}_{\tau}, \tag{3.20c}$$

$$\frac{\partial \bar{\alpha}_{\tau}}{\partial t} \zeta(\underline{\alpha}_{\tau}, \underline{\chi}_{\tau}; \frac{\partial \bar{\boldsymbol{\Pi}}_{\tau}}{\partial t} + (\bar{\mathbf{v}}_{\tau} \cdot \nabla) \bar{\boldsymbol{\Pi}}_{\tau}, \frac{\partial \bar{\alpha}_{\tau}}{\partial t} + (\bar{\mathbf{v}}_{\tau} \cdot \nabla) \bar{\alpha}_{\tau}) + \sqrt{\tau} \frac{\partial \bar{\alpha}_{\tau}}{\partial t} + \partial_{\alpha} \varphi(\bar{\mathbf{E}}_{\tau}, \bar{\alpha}_{\tau}, \bar{\chi}_{\tau}) \ni k_{\text{a}} \Delta \bar{\alpha}_{\tau}, \tag{3.20d}$$

$$\frac{\partial \bar{\chi}_{\tau}}{\partial t} + (\bar{\mathbf{v}}_{\tau} \cdot \nabla) \bar{\chi}_{\tau} = \operatorname{div} (\mathbb{M}(\underline{\alpha}_{\tau}, \underline{\chi}_{\tau}) \nabla \bar{\boldsymbol{\mu}}_{\tau}) \quad \text{with} \quad \bar{\boldsymbol{\mu}}_{\tau} = \partial_{\chi} \varphi(\bar{\mathbf{E}}_{\tau}, \bar{\alpha}_{\tau}, \bar{\chi}_{\tau}), \tag{3.20e}$$

and with the discrete structural stress

$$\bar{\mathbf{S}}_{\text{str}, \tau} = k_{\text{p}} \nabla \bar{\boldsymbol{\Pi}}_{\tau} \boxtimes \nabla \bar{\boldsymbol{\Pi}}_{\tau} + k_{\text{a}} \nabla \bar{\alpha}_{\tau} \boxtimes \nabla \bar{\alpha}_{\tau} - \left(\varphi(\bar{\mathbf{E}}_{\tau}, \bar{\alpha}_{\tau}, \bar{\chi}_{\tau}) + \frac{k_{\text{p}}}{2} |\nabla \bar{\boldsymbol{\Pi}}_{\tau}|^2 + \frac{k_{\text{a}}}{2} |\nabla \bar{\alpha}_{\tau}|^2 \right) \mathbf{I} \tag{3.20f}$$

and with the boundary conditions (3.6) written analogously. Actually, like (3.4c), the inclusions (3.20c,d) mean

$$\begin{aligned}
& \int_0^T \int_{\Omega} \zeta(\underline{\alpha}_\tau, \underline{\chi}_\tau; \tilde{\mathbf{H}}, \tilde{\alpha}) - \partial_{\mathbf{E}}\varphi(\bar{\mathbf{E}}_\tau, \bar{\alpha}_\tau, \bar{\chi}_\tau) : \left(\tilde{\mathbf{H}} - \frac{\partial \mathbf{\Pi}_\tau}{\partial t} - (\bar{\mathbf{v}}_\tau \cdot \nabla) \bar{\mathbf{\Pi}}_\tau \right) \\
& \quad + \left(\partial_{\alpha} \varphi(\bar{\mathbf{E}}_\tau, \bar{\alpha}_\tau, \bar{\chi}_\tau) + \sqrt{\tau} \frac{\partial \alpha_\tau}{\partial t} \right) \cdot \left(\tilde{\alpha} - \frac{\partial \alpha_\tau}{\partial t} - (\bar{\mathbf{v}}_\tau \cdot \nabla) \bar{\alpha}_\tau \right) + k_p \nabla \bar{\mathbf{\Pi}}_\tau : \nabla \tilde{\mathbf{H}} \\
& \quad + k_p \Delta \bar{\mathbf{\Pi}}_\tau : (\mathbf{v} \cdot \nabla) \bar{\mathbf{\Pi}}_\tau + k_a \nabla \bar{\alpha}_\tau : \nabla \tilde{\alpha} + k_a \Delta \bar{\alpha}_\tau \cdot (\mathbf{v} \cdot \nabla) \bar{\alpha}_\tau \, dx dt + \int_{\Omega} \frac{k_p}{2} |\nabla \mathbf{\Pi}_0|^2 + \frac{k_p}{2} |\nabla \alpha_0|^2 \, dx \\
& \geq \int_{\Omega} \frac{k_p}{2} |\nabla \mathbf{\Pi}_\tau(T)|^2 + \frac{k_p}{2} |\nabla \alpha_\tau(T)|^2 \, dx \\
& \quad + \int_0^T \int_{\Omega} \zeta(\underline{\alpha}_\tau, \underline{\chi}_\tau; \frac{\partial \mathbf{\Pi}_\tau}{\partial t} + (\bar{\mathbf{v}}_\tau \cdot \nabla) \bar{\mathbf{\Pi}}_\tau, \frac{\partial \alpha_\tau}{\partial t} + (\bar{\mathbf{v}}_\tau \cdot \nabla) \bar{\alpha}_\tau) \, dx dt
\end{aligned} \tag{3.21}$$

for any $(\tilde{\mathbf{H}}, \tilde{\alpha}) \in L^2(I; H^1(\Omega; \mathbb{R}^{d \times d} \times \mathbb{R}^\ell))$. Let us note that $\Delta \bar{\mathbf{\Pi}}_\tau \in L^2(I \times \Omega; \mathbb{R}^{d \times d})$ and $\Delta \bar{\alpha}_\tau \in L^2(I \times \Omega; \mathbb{R}^\ell)$, so that the integrals in (3.21) have a good sense. By putting $\tilde{\mathbf{H}} = 0$ and $\tilde{\alpha} = 0$, from (3.21) we can also read

$$\begin{aligned}
& \int_0^T \int_{\Omega} \partial_{\mathbf{E}}\varphi(\bar{\mathbf{E}}_\tau, \bar{\alpha}_\tau, \bar{\chi}_\tau) : \left(\frac{\partial \mathbf{\Pi}_\tau}{\partial t} + (\bar{\mathbf{v}}_\tau \cdot \nabla) \bar{\mathbf{\Pi}}_\tau \right) - \left(\partial_{\alpha} \varphi(\bar{\mathbf{E}}_\tau, \bar{\alpha}_\tau, \bar{\chi}_\tau) + \sqrt{\tau} \frac{\partial \alpha_\tau}{\partial t} \right) \cdot \left(\frac{\partial \alpha_\tau}{\partial t} + (\bar{\mathbf{v}}_\tau \cdot \nabla) \bar{\alpha}_\tau \right) \\
& \quad + k_p \Delta \bar{\mathbf{\Pi}}_\tau : (\mathbf{v} \cdot \nabla) \bar{\mathbf{\Pi}}_\tau + k_a \Delta \bar{\alpha}_\tau \cdot (\mathbf{v} \cdot \nabla) \bar{\alpha}_\tau \, dx dt + \int_{\Omega} \frac{k_p}{2} |\nabla \mathbf{\Pi}_0|^2 + \frac{k_p}{2} |\nabla \alpha_0|^2 \, dx \\
& \geq \int_0^T \int_{\Omega} \zeta(\underline{\alpha}_\tau, \underline{\chi}_\tau; \frac{\partial \mathbf{\Pi}_\tau}{\partial t} + (\bar{\mathbf{v}}_\tau \cdot \nabla) \bar{\mathbf{\Pi}}_\tau, \frac{\partial \alpha_\tau}{\partial t} + (\bar{\mathbf{v}}_\tau \cdot \nabla) \bar{\alpha}_\tau) \, dx dt + \int_{\Omega} \frac{k_p}{2} |\nabla \mathbf{\Pi}_\tau(T)|^2 + \frac{k_p}{2} |\nabla \alpha_\tau(T)|^2 \, dx.
\end{aligned} \tag{3.22}$$

Of course, we can write the above estimates on $[0, k\tau]$ with $k = 1, \dots, T/\tau$ instead of $I = [0, T]$. Altogether, we obtain a discrete energy-like balance

$$\begin{aligned}
& \int_{\Omega} \frac{\rho}{2} |\mathbf{v}_\tau(t)|^2 + \varphi(\mathbf{E}_\tau(t), \alpha_\tau(t), \chi_\tau(t)) + \frac{k_p}{2} |\nabla \mathbf{\Pi}_\tau(t)|^2 + \frac{k_a}{2} |\nabla \alpha_\tau(t)|^2 \, dx \\
& \quad + \int_0^t \left(\int_{\Omega} \zeta(\underline{\alpha}_\tau, \underline{\chi}_\tau; \frac{\partial \mathbf{\Pi}_\tau}{\partial t} + (\bar{\mathbf{v}}_\tau \cdot \nabla) \bar{\mathbf{\Pi}}_\tau, \frac{\partial \alpha_\tau}{\partial t} + (\bar{\mathbf{v}}_\tau \cdot \nabla) \bar{\alpha}_\tau) + k_v |\mathbf{e}(\bar{\mathbf{v}}_\tau)|^2 \right. \\
& \quad \left. + \mathbb{M}(\underline{\alpha}_\tau, \underline{\chi}_\tau) \nabla \bar{\mu}_\tau \cdot \nabla \bar{\mu}_\tau + k_e |\nabla \bar{\mathbf{S}}_\tau|^2 + \sqrt{\tau} \left| \frac{\partial \alpha_\tau}{\partial t} \right|^2 \, dx + \int_{\Gamma} \gamma |\bar{\mathbf{v}}_{t,\tau}|^2 \right. \\
& \quad \left. \leq \int_0^t \left(\int_{\Omega} \bar{\mathbf{f}}_\tau \cdot \bar{\mathbf{v}}_\tau + \frac{\sqrt{\tau}}{2} \left| \frac{\partial \alpha_\tau}{\partial t} + (\bar{\mathbf{v}}_\tau \cdot \nabla) \bar{\alpha}_\tau \right|^2 \, dx + \int_{\Gamma} \bar{\mathbf{g}}_{t,\tau} \cdot \bar{\mathbf{v}}_{t,\tau} + h \bar{\mu}_\tau \, dS \right) dt \right. \\
& \quad \left. + \int_{\Omega} \frac{\rho}{2} |\mathbf{v}_0|^2 + \varphi(\mathbf{E}_0, \alpha_0, \chi_0) + \frac{k_a}{2} |\nabla \alpha_0|^2 \, dx
\end{aligned} \tag{3.23}$$

for any $t = k\tau$. It should be emphasized that, as (3.23) involves the dissipation potential ζ and not the dissipation rate $\dot{\zeta}$, it is not a direct discrete analog of the energy balance (2.12), but it is sufficient for the a priori estimates. In fact, refining the argumentation, (3.23) with ξ could have been proved, too.

From the energetic inequality (3.23) by using the Young inequality for estimating $\int_{\Omega} \mathbf{f}_\tau^k \cdot \mathbf{v}_\tau^k \, dx \leq \|\mathbf{f}_\tau^k\|_{L^2(\Omega; \mathbb{R}^d)} \|\mathbf{v}_\tau^k\|_{L^2(\Omega; \mathbb{R}^d)} \leq \|\mathbf{f}_\tau^k\|_{L^2(\Omega; \mathbb{R}^d)} (1 + \|\mathbf{v}_\tau^k\|_{L^2(\Omega; \mathbb{R}^d)}^2)$ and by using the discrete Gronwall inequality, we obtain the following a priori estimates:

$$\|\bar{\mathbf{v}}_\tau\|_{L^\infty(I; L^2(\Omega; \mathbb{R}^d)) \cap L^2(I; H^1(\Omega; \mathbb{R}^d))} \leq C, \tag{3.24a}$$

$$\|\bar{\mathbf{E}}_\tau\|_{L^\infty(I; L^2(\Omega; \mathbb{R}^{d \times d}))} \leq C \quad \text{and} \quad \|\bar{\mathbf{S}}_\tau\|_{L^2(I; H^1(\Omega; \mathbb{R}^{d \times d}))} \leq C, \tag{3.24b}$$

$$\|\bar{\mathbf{\Pi}}_\tau\|_{L^\infty(I; H^1(\Omega; \mathbb{R}^{d \times d}))} \leq C \quad \text{and} \quad \left\| \frac{\partial \mathbf{\Pi}_\tau}{\partial t} + (\bar{\mathbf{v}}_\tau \cdot \nabla) \bar{\mathbf{\Pi}}_\tau \right\|_{L^2(I \times \Omega; \mathbb{R}^{d \times d})} \leq C, \tag{3.24c}$$

$$\|\bar{\alpha}_\tau\|_{L^\infty(I; H^1(\Omega; \mathbb{R}^\ell))} \leq C, \quad \left\| \frac{\partial \alpha_\tau}{\partial t} + (\bar{\mathbf{v}}_\tau \cdot \nabla) \bar{\alpha}_\tau \right\|_{L^2(I \times \Omega; \mathbb{R}^\ell)} \leq C, \quad \text{and} \quad \left\| \frac{\partial \alpha_\tau}{\partial t} \right\|_{L^2(I \times \Omega; \mathbb{R}^\ell)} \leq \frac{C}{\sqrt[4]{\tau}}, \tag{3.24d}$$

$$\|\bar{\chi}_\tau\|_{L^\infty(I; L^2(\Omega))} \leq C \quad \text{and} \quad \|\bar{\mu}_\tau\|_{L^2(I; H^1(\Omega))} \leq C. \tag{3.24e}$$

Actually, the estimate (3.24a) is due to the Korn inequality. Moreover, from the calculus

$$\begin{aligned}
\nabla \mathbf{S} &= \nabla \partial_{\mathbf{E}}\varphi(\mathbf{E}, \alpha, \chi) = \partial_{\mathbf{E}\mathbf{E}}^2\varphi(\mathbf{E}, \alpha, \chi) \nabla \mathbf{E} + \partial_{\mathbf{E}\alpha}^2\varphi(\mathbf{E}, \alpha, \chi) \nabla \alpha + \partial_{\mathbf{E}\chi}^2\varphi(\mathbf{E}, \alpha, \chi) \nabla \chi \quad \text{and} \\
\nabla \mu &= \nabla \partial_{\chi}\varphi(\mathbf{E}, \alpha, \chi) = \partial_{\mathbf{E}\chi}^2\varphi(\mathbf{E}, \alpha, \chi) \nabla \mathbf{E} + \partial_{\alpha\chi}^2\varphi(\mathbf{E}, \alpha, \chi) \nabla \alpha + \partial_{\chi\chi}^2\varphi(\mathbf{E}, \alpha, \chi) \nabla \chi,
\end{aligned}$$

we can see

$$\begin{pmatrix} \nabla \bar{\mathbf{E}}_\tau \\ \nabla \bar{\chi}_\tau \end{pmatrix} = \begin{pmatrix} \partial_{\mathbf{E}\mathbf{E}}^2 \varphi(\bar{\mathbf{E}}_\tau, \bar{\alpha}_\tau, \bar{\chi}_\tau) & \partial_{\mathbf{E}\chi}^2 \varphi(\bar{\mathbf{E}}_\tau, \bar{\alpha}_\tau, \bar{\chi}_\tau) \\ \partial_{\mathbf{E}\chi}^2 \varphi(\bar{\mathbf{E}}_\tau, \bar{\alpha}_\tau, \bar{\chi}_\tau) & \partial_{\chi\chi}^2 \varphi(\bar{\mathbf{E}}_\tau, \bar{\alpha}_\tau, \bar{\chi}_\tau) \end{pmatrix}^{-1} \begin{pmatrix} \nabla \bar{\mathbf{S}}_\tau - \partial_{\mathbf{E}\alpha}^2 \varphi(\bar{\mathbf{E}}_\tau, \bar{\alpha}_\tau, \bar{\chi}_\tau) \nabla \bar{\alpha}_\tau \\ \nabla \bar{\mu}_\tau - \partial_{\alpha\chi}^2 \varphi(\bar{\mathbf{E}}_\tau, \bar{\alpha}_\tau, \bar{\chi}_\tau) \nabla \bar{\alpha}_\tau \end{pmatrix}.$$

From this, by using also the assumption (3.2c) which implies boundedness of the inverse of the Hessian $\partial_{(\mathbf{E},\chi),(\mathbf{E},\chi)}^2 \varphi$, we can still read the estimate

$$\|\bar{\mathbf{E}}_\tau\|_{L^2(I;H^1(\Omega;\mathbb{R}^{d\times d}))} \leq C \quad \text{and} \quad \|\bar{\chi}_\tau\|_{L^2(I;H^1(\Omega))} \leq C. \quad (3.24f)$$

From the $L^\infty(I;L^2(\Omega))$ -estimates of the gradients of $\bar{\mathbf{\Pi}}_\tau$ and $\bar{\alpha}_\tau$ (3.24c,d), we can then estimate also

$$\left\| \frac{\partial \bar{\mathbf{\Pi}}_\tau}{\partial t} \right\|_{L^{4/3}(I\times\Omega;\mathbb{R}^{d\times d})} \leq C, \quad (3.25a)$$

$$\left\| \frac{\partial \bar{\alpha}_\tau}{\partial t} \right\|_{L^{4/3}(I\times\Omega;\mathbb{R}^\ell)} \leq C, \quad (3.25b)$$

$$\begin{aligned} \left\| \frac{\partial \bar{\chi}_\tau}{\partial t} \right\|_{L^2(I;H^1(\Omega)^*)} &= \sup_{\|\tilde{\mu}\|_{L^2(I;H^1(\Omega))} \leq 1} \int_0^T \left(\int_\Omega (\mathbb{M}(\alpha_\tau, \chi_\tau) \nabla \bar{\mu}_\tau - \bar{\mathbf{v}}_\tau) \cdot \nabla \tilde{\mu} \right. \\ &\quad \left. - (\operatorname{div} \bar{\mathbf{v}}_\tau) \bar{\chi}_\tau \tilde{\mu} \, dx + \int_\Gamma h \tilde{\mu} \, dS \right) dt \leq C. \end{aligned} \quad (3.25c)$$

For (3.25a,b), we used $\bar{\mathbf{v}}_\tau \in L^\infty(I;L^2(\Omega;\mathbb{R}^d)) \cap L^2(I;H^1(\Omega;\mathbb{R}^d)) \subset L^4(I\times\Omega;\mathbb{R}^d)$ so that certainly $(\bar{\mathbf{v}}_\tau \cdot \nabla) \bar{\mathbf{\Pi}}_\tau \in L^{4/3}(I\times\Omega;\mathbb{R}^{d\times d})$ and $(\bar{\mathbf{v}}_\tau \cdot \nabla) \bar{\alpha}_\tau \in L^{4/3}(I\times\Omega;\mathbb{R}^\ell)$. Moreover, by $\frac{\partial}{\partial t} \mathbf{E}_\tau = \mathbf{e}(\bar{\mathbf{v}}_\tau) - \frac{\partial}{\partial t} \mathbf{\Pi}_\tau - (\bar{\mathbf{v}}_\tau \cdot \nabla) \bar{\mathbf{\Pi}}_\tau + k_e \Delta \bar{\mathbf{S}}_\tau - (\bar{\mathbf{v}}_\tau \cdot \nabla) \bar{\mathbf{E}}_\tau$, cf. (3.20b), we have also

$$\left\| \frac{\partial \bar{\mathbf{E}}_\tau}{\partial t} \right\|_{L^2(I;H^1(\Omega;\mathbb{R}^{d\times d})^*) + L^{4/3}(I\times\Omega;\mathbb{R}^{d\times d})} \leq C. \quad (3.25d)$$

By comparison

$$\frac{\partial \bar{\mathbf{v}}_\tau}{\partial t} = \frac{\operatorname{div}(\bar{\mathbf{S}}_\tau + \bar{\mathbf{S}}_{\text{str},\tau} + k_v \mathbf{e}(\bar{\mathbf{v}}_\tau) + \bar{\mathbf{f}}_\tau)}{\varrho} - (\bar{\mathbf{v}}_\tau \cdot \nabla) \bar{\mathbf{v}}_\tau - \frac{1}{2} (\operatorname{div} \bar{\mathbf{v}}_\tau) \bar{\mathbf{v}}_\tau$$

with $\bar{\mathbf{S}}_{\text{str},\tau}$ the piecewise constant interpolant of the structural stress, cf. (3.20a) and (3.20f), we have also

$$\left\| \frac{\partial \bar{\mathbf{v}}_\tau}{\partial t} \right\|_{L^2(I;H^3(\Omega;\mathbb{R}^d)^*)} \leq C. \quad (3.25e)$$

Here we used that, by (3.24c), $\nabla \bar{\mathbf{\Pi}}_\tau \boxtimes \nabla \bar{\mathbf{\Pi}}_\tau - \frac{1}{2} |\nabla \bar{\mathbf{\Pi}}_\tau|^2 \mathbf{I} \in L^\infty(I;L^1(\Omega;\mathbb{R}^{d\times d}))$ and similarly, by (3.24d), also $\nabla \bar{\alpha}_\tau \boxtimes \nabla \bar{\alpha}_\tau - \frac{1}{2} |\nabla \bar{\alpha}_\tau|^2 \mathbf{I} \in L^\infty(I;L^1(\Omega;\mathbb{R}^{d\times d}))$, and also that ϱ is assumed constant. Also, for the limit passage in (3.21), we need the estimates

$$\|\Delta \bar{\mathbf{\Pi}}_\tau\|_{L^2(I\times\Omega;\mathbb{R}^{d\times d})} \leq C \quad \text{and} \quad \|\Delta \bar{\alpha}_\tau\|_{L^2(I\times\Omega;\mathbb{R}^\ell)} \leq C, \quad (3.25f)$$

which can be seen by comparison from (3.20c,d).

Step 4. (Convergence): By the Banach selection principle, we obtain a subsequence converging weakly* with respect to topologies indicated in (3.24) and (3.25). Moreover, we now prove also the strong convergence

$$\nabla \bar{\mathbf{\Pi}}_\tau \rightarrow \nabla \mathbf{\Pi} \quad \text{strongly in } L^2(I\times\Omega;\mathbb{R}^{d\times d\times d}) \quad \text{and} \quad (3.26a)$$

$$\nabla \bar{\alpha}_\tau \rightarrow \nabla \alpha \quad \text{strongly in } L^2(I\times\Omega;\mathbb{R}^{d\times \ell}). \quad (3.26b)$$

To prove it, we take sequences $\{\tilde{\mathbf{\Pi}}_\tau\}_{\tau>0}$ and $\{\tilde{\alpha}_\tau\}_{\tau>0}$ piecewise constant in time with respect to the partition with the time step τ and, for $\tau \rightarrow 0$, converging strongly towards $\mathbf{\Pi}$ and α , respectively. Then we can see that

$$\begin{aligned} \int_0^T \int_\Omega k_p |\nabla(\bar{\mathbf{\Pi}}_\tau - \tilde{\mathbf{\Pi}}_\tau)|^2 \, dx dt &= - \int_0^T \int_\Omega \left(\partial_{\frac{\mathbf{D}\mathbf{\Pi}}{\partial t}} \zeta(\alpha_\tau, \chi_\tau; \frac{\partial \bar{\mathbf{\Pi}}_\tau}{\partial t} + (\bar{\mathbf{v}}_\tau \cdot \nabla) \bar{\mathbf{\Pi}}_\tau, \frac{\partial \bar{\alpha}_\tau}{\partial t} + (\bar{\mathbf{v}}_\tau \cdot \nabla) \bar{\alpha}_\tau) \right. \\ &\quad \left. + \bar{\mathbf{S}}_\tau \right) : (\bar{\mathbf{\Pi}}_\tau - \tilde{\mathbf{\Pi}}_\tau) + k_p \nabla \tilde{\mathbf{\Pi}}_\tau : \nabla(\bar{\mathbf{\Pi}}_\tau - \tilde{\mathbf{\Pi}}_\tau) \, dx dt \rightarrow 0 \end{aligned} \quad (3.27a)$$

and similarly

$$\begin{aligned} \int_0^T \int_{\Omega} k_a |\nabla(\bar{\alpha}_\tau - \tilde{\alpha}_\tau)|^2 dxdt &= - \int_0^T \int_{\Omega} \left(\frac{\partial \bar{\Pi}_\tau}{\partial t} \zeta(\underline{\alpha}_\tau, \underline{\chi}_\tau; \frac{\partial \bar{\Pi}_\tau}{\partial t} + (\bar{\mathbf{v}}_\tau \cdot \nabla) \bar{\Pi}_\tau, \frac{\partial \alpha_\tau}{\partial t} + (\bar{\mathbf{v}}_\tau \cdot \nabla) \alpha_\tau) \right. \\ &\quad \left. + \partial_\alpha \varphi(\bar{\mathbf{E}}_\tau, \bar{\alpha}_\tau, \bar{\chi}_\tau) + \sqrt{\tau} \frac{\partial \alpha_\tau}{\partial t} \right) \cdot (\bar{\alpha}_\tau - \tilde{\alpha}_\tau) + k_a \nabla \tilde{\alpha}_\tau : \nabla(\bar{\alpha}_\tau - \tilde{\alpha}_\tau) dxdt \rightarrow 0. \end{aligned} \quad (3.27b)$$

Here we used (3.2c) so that $\partial_{\mathbf{E}} \varphi(\bar{\mathbf{E}}_\tau, \bar{\alpha}_\tau, \bar{\chi}_\tau)$ and $\partial_\alpha \varphi(\bar{\mathbf{E}}_\tau, \bar{\alpha}_\tau, \bar{\chi}_\tau)$ are bounded in the respective $L^{6/5+\epsilon}(I \times \Omega)$ -spaces while $\bar{\Pi}_\tau - \tilde{\Pi}_\tau \rightarrow 0$ and $\bar{\alpha}_\tau - \tilde{\alpha}_\tau \rightarrow 0$ strongly in $L^{6-\epsilon}(I \times \Omega; \mathbb{R}^{d \times d})$ and $L^{6-\epsilon}(I \times \Omega; \mathbb{R}^\ell)$, respectively; this is due to the Aubin-Lions theorem, relying on (3.24e) with (3.25b). In (3.27a), we used that $\frac{\partial \bar{\Pi}_\tau}{\partial t} \zeta(\underline{\alpha}_\tau, \underline{\chi}_\tau; \frac{\partial \bar{\Pi}_\tau}{\partial t} + (\bar{\mathbf{v}}_\tau \cdot \nabla) \bar{\Pi}_\tau, \frac{\partial \alpha_\tau}{\partial t} + (\bar{\mathbf{v}}_\tau \cdot \nabla) \alpha_\tau)$ is bounded in $L^2(I \times \Omega; \mathbb{R}^{d \times d})$. Similarly, in (3.27b), we used that $\frac{\partial \bar{\Pi}_\tau}{\partial t} \zeta(\underline{\alpha}_\tau, \underline{\chi}_\tau; \frac{\partial \bar{\Pi}_\tau}{\partial t} + (\bar{\mathbf{v}}_\tau \cdot \nabla) \bar{\Pi}_\tau, \frac{\partial \alpha_\tau}{\partial t} + (\bar{\mathbf{v}}_\tau \cdot \nabla) \alpha_\tau)$ is bounded in $L^2(I \times \Omega; \mathbb{R}^\ell)$ and, moreover, that $\|\sqrt{\tau} \frac{\partial \alpha_\tau}{\partial t}\|_{L^2(I \times \Omega; \mathbb{R}^\ell)} = \mathcal{O}(\sqrt[4]{\tau}) \rightarrow 0$ due to the last estimate in (3.24d).

Based on the estimates (3.24f) and (3.25d), we have $\bar{\mathbf{E}}_\tau \rightarrow \mathbf{E}$ strongly in $L^2(I; L^{6-\epsilon}(\Omega; \mathbb{R}^{d \times d}))$ due to the Aubin-Lions theorem, generalized for functions whose time-derivatives are measures as in [30, Cor.7.9]. By the interpolation with the estimate in $L^\infty(I; L^2(\Omega; \mathbb{R}^{d \times d}))$, we have the strong convergence even in a smaller space, e.g. in $L^4(I; L^{3-\epsilon}(\Omega; \mathbb{R}^{d \times d}))$. Thanks to the growth condition (3.2a) from which we have also $|\varphi(\mathbf{E}, \alpha, \chi)| \leq (1 + |\mathbf{E}|^{5/2-\epsilon} + |\alpha|^{4-\epsilon} + |\chi|^{4-\epsilon})/\epsilon$, we can see that $\varphi(\bar{\mathbf{E}}_\tau, \bar{\alpha}_\tau, \bar{\chi}_\tau)$ converges strongly in $L^{6/5-\epsilon}(I \times \Omega)$. Taking into account also (3.27), we obtain the convergence in the structural stress (3.20f), namely $\bar{\mathbf{S}}_{\text{str}, \tau} \rightarrow \mathbf{S}_{\text{str}}$ strongly in $L^1(I \times \Omega; \mathbb{R}^{d \times d})$ with \mathbf{S}_{str} from (2.3). Thus, noting that $\bar{\mathbf{S}}_\tau = \partial_{\mathbf{E}} \varphi(\bar{\mathbf{E}}_\tau, \bar{\alpha}_\tau, \bar{\chi}_\tau)$ converges even strongly in $L^2(I \times \Omega; \mathbb{R}^{d \times d})$ due to the growth condition (3.2a), we can pass to the limit in the momentum equation (3.20a). The limit passage in (3.20b) is similar.

By the proved strong convergence of $\bar{\mathbf{E}}_\tau \rightarrow \mathbf{E}$, we can pass to the limit in the nonlinear terms $\partial_{\mathbf{E}} \varphi(\bar{\mathbf{E}}_\tau, \bar{\alpha}_\tau, \bar{\chi}_\tau)$ and $\partial_\alpha \varphi(\bar{\mathbf{E}}_\tau, \bar{\alpha}_\tau, \bar{\chi}_\tau)$ flow rule, i.e. in the variational inequality (3.21), and in the terms $\partial_\chi \varphi(\bar{\mathbf{E}}_\tau, \bar{\alpha}_\tau, \bar{\chi}_\tau)$ and $\mathbb{M}(\underline{\alpha}_\tau, \underline{\chi}_\tau)$ in the diffusion equation (3.20e), too.

Let us also note that the convexifying term in (3.20d) vanishes in the limit due to the estimate (3.25b) because obviously $\|\sqrt{\tau} \frac{\partial \alpha_\tau}{\partial t}\|_{L^2(I \times \Omega; \mathbb{R}^\ell)} = \mathcal{O}(\sqrt[4]{\tau}) \rightarrow 0$, as used already before in (3.27b).

Eventually, from (3.25f) and (3.24f), we also obtain the $L^2(I \times \Omega)$ -information about $\Delta \bar{\Pi}$, $\Delta \alpha$, and $\nabla \chi$. \square

4 Concluding remarks

We close this paper with several remarks, outlining some concrete examples, expansions, or comments to the used analysis.

Remark 4.1 (Example for a semi-convex φ .) The so-called (weakened) semi-convexity (3.2b) is not in conflict with usual damage models and, when combined with *Biot's poroelasticity*, it allows for models like

$$\varphi(\mathbf{E}, \alpha, \chi) = \frac{dK}{2} |\text{sph} \mathbf{E}|^2 + \frac{M}{2} |\beta \text{tr} \mathbf{E} - \chi + \chi_{\text{eq}}|^2 + G(\alpha) \frac{|\text{dev} \mathbf{E}|^2}{1 + \epsilon |\text{dev} \mathbf{E}|^2} + G_0 |\text{dev} \mathbf{E}|^2 + \phi(\alpha), \quad (4.1)$$

where χ_{eq} is a given equilibrium concentration, “sph” denotes the spherical part (recall that $\text{sph} \mathbf{E} = \mathbf{E} - \text{dev} \mathbf{E} = (\text{tr} \mathbf{E}) \mathbf{I} / d$) with K the bulk modulus, “tr” denotes the trace, and “dev” the deviatoric part with the shear modulus $G : \mathbb{R}^\ell \rightarrow \mathbb{R}$ non-negative smooth satisfying $G'_i(\dots, 0, \dots) = 0 = G'_i(\dots, 1, \dots)$ for $i = 1, \dots, \ell$, which ensures that each α_i takes values in the interval $[0, 1]$ as usually requires in damage/breakage type models. Further parameters K , M , and β in (4.1) have the meaning of the bulk modulus, Biot's modulus, and Biot coefficient, respectively, while $G_0 > 0$ is just small regularizing modulus not subjected to damage and ensuring coercivity. This is the classical Biot model for a saturated fluid flow in poroelastic media [4]. Note that the second derivatives of the $G(\alpha)$ -term are bounded so that (3.2a) holds. For a convexification by a quadratic form in (\mathbf{E}, χ) see [32] which deals with a non-convective variant and which would be here more difficult. Actually, the Biot ansatz (4.1) gives the chemical potential $\mu = M(\beta \text{tr} \mathbf{E} - \chi)$, meaning a pressure and then the flux in the Fick diffusion turns rather to the *Darcy law*. The last term in (4.1) creates a driving force for healing of damage. Together with the $\Delta \alpha$ in the damage flow-rule (2.1d), it enables to model the Ambrosio-Tortorelli-type *phase-field fracture*; actually, the standard choice is $G(\alpha) = G_1 \alpha^2$, $G_0 = k_a^2$ and $\phi(\alpha) = (1 - \alpha)^2 / (2k_a)$ with $k_a > 0$ from (2.1d) assumed small.

Remark 4.2 (Energy conservation.) The energy balance (2.12) is only formal and its rigorous proof needs to legitimate the test used in (2.5a)–(2.11). This does not seem easily possible, however, and a regularization of the model seems necessary. More specifically, a higher-order viscosity of the type $\operatorname{div}^2(k_v |\nabla \mathbf{e}(\mathbf{v})|^{p-2} \nabla \mathbf{e}(\mathbf{v}))$ for $p > d$ together with the viscous variant of the diffusion $\mu = \partial_\chi \varphi(\mathbf{E}, \alpha, \chi) + \epsilon \frac{D}{Dt} \chi$ with some (presumably small) modulus $\epsilon > 0$ (with the physical dimension $\text{Pa s} = \text{J s/m}^3$) would help, cf. [34] or [35, Sect.8]. This would only make some arguments a bit more complicated and open a possibility for an expansion of the model towards full thermodynamics by completing it by the heat-transfer equation. As for the analysis, first the limit passage in the mechanical part using also the strong convergence of temperature by the Aubin-Lions compactness theorem should be done, followed by the strong convergence of the dissipation rate, and finished by the convergence in the heat-transfer equation. We refer to [17, Chap.8] or also e.g. [34] for the technical details.

Remark 4.3 (Staggered time discretisation.) One could think about a fractional-step splitting (also known as a staggered) time discretisation to decouple $(\mathbf{v}_\tau^k, \mathbf{E}_\tau^k, \mathbf{\Pi}_\tau^k)$ from α_τ^k and from $(\chi_\tau^k, \mu_\tau^k)$ in order to allow for a separately convex φ . This usually works efficiently, although here it would lead to a coupled scheme through the structural stress but, more important, here there would be troubles with modification of the calculus (3.17). This is the reason that we used the fully implicit time discretisation (3.5).

Remark 4.4 (Galerkin method.) In our convective model, the Galerkin approximation (i.e. the space discretisation instead of the time discretisation (3.5)) would face serious technical difficulties because testing by convective time derivatives which do not comply with finite-dimensional spaces used for the Galerkin method and the sophisticated calculus like (3.13)–(3.14) or (3.17) would not be legal. Therefore, the implementation of this, usually very efficient technique seems problematic here.

Remark 4.5 (More general stored energies.) The stored energy φ is often considered not convex in geophysical applications, as devised in [23] and used e.g. in [20,21,22]. This brings, however, technical difficulties in analysis. In particular, it violates the assumption (3.2c) which is needed to control $\nabla \mathbf{E}$ which was used to obtain strong convergence in \mathbf{E} . And this strong is needed to pass to the limit in the nonlinear terms $\varphi(\cdot, \alpha, \chi) \mathbf{I}$ and $\partial_{\mathbf{E}} \varphi(\cdot, \alpha, \chi)$ in particular in such a nonconvex situation.

Remark 4.6 (Other phenomena involved.) The Eulerian description opens a way for enhancement of the model by other phenomena which ultimately needs formulation in Eulerian configuration. In particular, it concerns gravity and magnetic fields. Also, a coupling with fluidic regions (in particular with the outer core of the Earth) is thus well facilitated.

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