Meccanica **30:** 417–424, 1995.

CLASSICAL TOPICS REVISITED

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The Editor

Mohr's Arbelos

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(Received: 1 July 1994; accepted in revised form: 7 December 1994)

Abstract. The classical representation of the stress state due to O. Mohr is obtained by using the calculus of constrained extrema.

Sommario. Si ottiene la classica rappresentazione di O. Mohr dello stato tensionale con i metodi del calcolo di estremi condizionati.

Key words: Mohr's circles, Stress representation, Solid mechanics.

1. Introduction

For f a deformation of a continuous body Ω, the Cauchy stress $\mathbf{S}$ is a field over $f(\Omega)$, the deformed configuration of Ω under f, whose values are symmetric second-order tensors. At a point p of $f(\Omega)$, for each oriented surface $\mathcal{S}_{p,\mathbf{n}}$ through p whose normal unit vector at p is $\mathbf{n}$, the construct

$$\mathbf{s}(p,\mathbf{n}) = \mathbf{S}(p)\mathbf{n} \tag{1.1}$$

is called the *stress vector*, and is interpreted as the contact force, per unit area, exerted across $\mathcal{S}_{p,\mathbf{n}}$ upon the material on the negative side of $\mathcal{S}_{p,\mathbf{n}}$ by the material on the positive side.

Let Ω, f, $p \in f(\Omega)$, and $\mathbf{S}(p)$ be fixed in the following discussion, and let the stress vector relative to a plane of normal $\mathbf{n}$ be resolved into two vectors, the one orthogonal, the other parallel to that plane:

$$(\mathbf{n}\otimes\mathbf{n})\mathbf{S}\mathbf{n} = (\mathbf{n}\cdot\mathbf{S}\mathbf{n})\mathbf{n}, \quad (\mathbf{I}-\mathbf{n}\otimes\mathbf{n})\mathbf{S}\mathbf{n} = \mathbf{S}\mathbf{n} - (\mathbf{n}\cdot\mathbf{S}\mathbf{n})\mathbf{n} \tag{1.2}$$

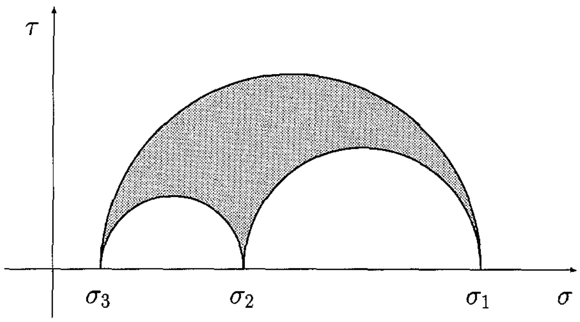

Fig. 1. The range of the tension mapping.

(here $\cdot$ and $\otimes$ denote the scalar and the tensor product, respectively, and $\mathbf{n}\otimes\mathbf{n}$, $(\mathbf{I}-\mathbf{n}\otimes\mathbf{n})$ are the complementary orthogonal projectors relative to the direction $\mathbf{n}$). The first vector is called the *normal force*, the second the *shear force* (per unit area) on the plane oriented by $\mathbf{n}$;

$$\sigma(\mathbf{n}) := \mathbf{n}\cdot\mathbf{Sn}, \tag{1.3}$$

the length of the normal force vector, is the *normal tension* in the direction $\mathbf{n}$, and

$$\tau(\mathbf{n}) := \|\, \mathbf{Sn} - (\mathbf{n}\cdot\mathbf{Sn})\mathbf{n} \,\| = \sqrt{\mathbf{S}^2\mathbf{n}\cdot\mathbf{n} - (\mathbf{Sn}\cdot\mathbf{n})^2}, \tag{1.4}$$

the length of the shear force vector, is the *tangential tension* on the plane perpendicular to $\mathbf{n}$; needless to say,

$$\sigma(\mathbf{n})^2 + \tau(\mathbf{n})^2 = \|\, \mathbf{Sn} \,\|^2 . \tag{1.5}$$

The *tension mapping*

$$\mathcal{T}(\mathbf{n}) := (\sigma(\mathbf{n}), \tau(\mathbf{n})) \tag{1.6}$$

maps $\mathcal{U}$, the unit sphere of the three-dimensional euclidean vector space $\mathcal{V}$, into $\mathbb{R}\times\mathbb{R}^+$. The aim of this presentation is to describe the range $\mathcal{T}(\mathcal{U})$ of the tension mapping.

This issue was first addressed by O. Mohr [1].[1] For $\sigma_1 \geq \sigma_2 \geq \sigma_3$ the proper values of $\mathbf{S}$,[2] Mohr proved that $\mathcal{T}(\mathcal{U})$ is the region in the half plane $\mathbb{R}\times\mathbb{R}^+$ bounded by three circles touching each other (called Mohr's circles after him), whose centres are on the σ-axis at abscissae $(\sigma_1+\sigma_3)/2$, $(\sigma_1+\sigma_2)/2$, $(\sigma_2+\sigma_3)/2$, and whose radii are $(\sigma_1-\sigma_3)/2$, $(\sigma_1-\sigma_2)/2$, $(\sigma_2-\sigma_3)/2$, respectively (Fig. 1).

Mohr's proof is flawless, but many find it rather long and involved; with minor variants, it is included in virtually all textbooks on Strength of Materials, and is currently taught in most of the relative undergraduate courses. Our approach is different from Mohr's, and so is our machinery: we reduce the representation of $\mathcal{T}(\mathcal{U})$ to a problem of constrained extrema, and use elementary calculus with a few bits of linear algebra. We believe that for engineering undergraduates ours might turn out to be an easier argument to follow than Mohr's; in any event, we also include a cleansed account of the standard treatments (Appendix A) and, for the

[1] See also Westergaard [2]. Closely related topics have been treated recently in [3], [4].

[2] Given a symmetric tensor $\mathbf{S}$, a *proper pair* $(\sigma, \mathbf{s})$ of $\mathbf{S}$ consists of a *proper value* $\sigma\in\mathbb{R}$ and a *proper vector* $\mathbf{s}$ of unit length, such that $\mathbf{Ss} = \sigma\mathbf{s}$.

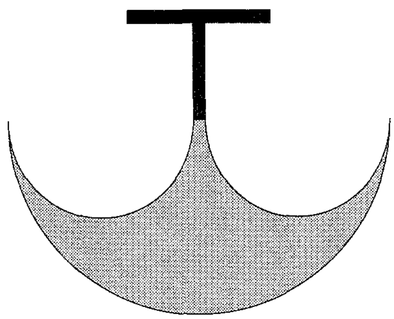

Fig. 2. A leatherworker's knife.

mathematically inclined reader, in Appendix B we outline a topological approach to Mohr's three circles.

Scholion. Mohr did not bother to give $\mathcal{T}(\mathcal{U})$ a name. Some Italian authors, apparently after Franciosi [5], refer to this region as *'Mohr's arbelos'*. [3] This curious name, arbelos, is not inappropriate.

The ἄρβηλος, notes Franciosi, is a leatherworker's knife (see. Fig. 2, a sketch from the figure on p. 76 of [7]), let him be a shoemaker, as the standard acceptation goes [8], [9], or a saddler, as authoritatively surmised by Thompson [7]: '... I once asked my London bootmaker, who told me ... that that was not a shoemaker's but a saddler's knife ... It is a semicircular blade, with a short handle fixed to its centre ... two lesser semicircles are cut out of the blade on either side of the handle ... they furnish two sharp points, to be used ... for piercing holes in a strap which the knife has cut.'

Apparently, the ἄρβηλος was a tool in common use among shoemakers or saddlers of Syracuse, Magna Graecia, already in the third century B.C.: indeed, the ἄρβηλος has a mathematical citizenship - we learn from Thomas [9] – because three propositions about it (e.g. a formula for computing its area) are contained in Archimedes' *Liber Assumptorum*; these propositions are included in Pappus' *Collection*, together with another attractive proposition about an infinite sequence of circles inscribed in an ἄρβηλος.

Thus, there are reasons to call Mohr's arbelos the region in the half plane $\mathbb{R} \times \mathbb{R}^+$ delimited by Mohr's circles. There is, however, a caveat: whenever two of the proper values of $\mathbf{S}$ are equal, Mohr's circles coalesce into one, so that, instead of a leatherworker's knife, one would be reminded instead of a peasant's sickle; worse than that, when the stress is an uniform pressure, Mohr's circles all shrink to one point, and we are left with no tool whatsoever. □

2. The range of the tension mapping

Consider the mappings $\sigma(\cdot)$ and $\tau(\cdot)$ defined by (1.3) and (1.4), respectively. The latter is nonnegative-valued; the former, as a consequence of the notion of proper pair for a symmetric tensor, satisfies the inequalities

$$\sigma_3 \leq \sigma(\mathbf{n}) \leq \sigma_1, \quad \mathbf{n} \in \mathcal{U}; \tag{2.1}$$

[3] *Cf., e.g.*, Di Tommaso [6], Romano and Romano [11].

hence, the range $\mathcal{T}(\mathcal{U})$ of the tension mapping lies in the half-strip $[\sigma_3, \sigma_1] \times \mathbb{R}^+$.

We choose to characterize $\mathcal{T}(\mathcal{U})$ by proving that, for each $\sigma \in [\sigma_3, \sigma_1]$, the collection of its points of abscissa σ is the segment having one endpoint on the curve which is the union of the inner Mohr's half circles, and the other endpoint on the outer Mohr's half circle.

Let us fix σ, define the set

$$\mathcal{U}_\sigma = \{\mathbf{v} \in \mathcal{V} \,|\, \alpha(\mathbf{v}) := \mathbf{v} \cdot \mathbf{v} - 1 = 0\,,\ \beta_\sigma(\mathbf{v}) := \mathbf{S}\mathbf{v} \cdot \mathbf{v} - \sigma = 0\}\,, \tag{2.2}$$

and introduce the mapping

$$\tau_\sigma(\mathbf{v}) := \|\, \mathbf{S}\mathbf{v} - (\mathbf{S}\mathbf{v} \cdot \mathbf{v})\mathbf{v} \,\| \tag{2.3}$$

from $\mathcal{U}_\sigma$ into $\mathbb{R}^+$. Since the set $\mathcal{U}_\sigma$ is closed and bounded, such is its image under the continuous mapping τ_σ; hence, τ_σ has both a global maximum and a global minimum on $\mathcal{U}_\sigma$. We shall show that

(i) τ_σ has one local maximum $\overline{\tau}_\sigma$ and one local minimum $\underline{\tau}_\sigma$,

so that the range of τ_σ is the interval $[\underline{\tau}_\sigma, \overline{\tau}_\sigma]$. We shall also show that

(ii) the extrema $\overline{\tau}_\sigma$ and $\underline{\tau}_\sigma$ have the expressions

$$\begin{aligned} \overline{\tau}_\sigma &= \sqrt{(\sigma_1 - \sigma)(\sigma - \sigma_3)}, \quad \sigma \in [\sigma_3, \sigma_1], \\ \underline{\tau}_\sigma &= \begin{cases} \sqrt{(\sigma_2 - \sigma)(\sigma - \sigma_3)}, & \sigma \in [\sigma_3, \sigma_2] \\ \sqrt{(\sigma_1 - \sigma)(\sigma - \sigma_2)}, & \sigma \in [\sigma_2, \sigma_1]. \end{cases} \end{aligned} \tag{2.4}$$

This will be enough to yield the announced characterization of $\mathcal{T}(\mathcal{U})$, since $(\sigma, \overline{\tau}_\sigma)$ is a typical point of the outer Mohr's half circle, while $(\sigma, \underline{\tau}_\sigma)$ is a typical point of the union of the inner Mohr's half circles.

For each $\sigma \in [\sigma_3, \sigma_1]$, we consider the mapping

$$\phi_\sigma(\mathbf{v}, \lambda, \mu) := \|\, \mathbf{S}\mathbf{v} - (\mathbf{v} \cdot \mathbf{S}\mathbf{v})\mathbf{v} \,\| \ + \mu\, \alpha(\mathbf{v}) + \lambda\, \beta_\sigma(\mathbf{v}), \tag{2.5}$$

from $\mathcal{V} \times \mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. A basic result from the calculus of constrained extrema [10] is that one of the following situations has to occur when τ_σ attains a local extremum at $\bar{\mathbf{v}} \in \mathcal{U}_\sigma$:

(a) there are Lagrange multipliers $\bar{\lambda}$ and $\bar{\mu}$ such that ϕ_σ is stationary at $(\bar{\mathbf{v}}, \bar{\lambda}, \bar{\mu})$;

(b) the gradient vectors of α and β_σ at $\bar{\mathbf{v}}$ are parallel, i.e.

$$\bar{\mathbf{v}} \times \mathbf{S}\bar{\mathbf{v}} = \mathbf{0} \tag{2.6}$$

(here $\times$ denotes the vector product);

(c) the mapping $\phi_\sigma(\cdot, \lambda, \mu)$ is not differentiable at $\bar{\mathbf{v}}$, i.e.

$$\|\, \mathbf{S}\bar{\mathbf{v}} - (\bar{\mathbf{v}} \cdot \mathbf{S}\bar{\mathbf{v}})\bar{\mathbf{v}} \,\| = 0. \tag{2.7}$$

With a view toward exploiting the implications of this result, we first note that $\bar{\mathbf{v}} \in \mathcal{U}_\sigma$ solves (2.6) (or (2.7)) if and only if $\bar{\mathbf{v}}$ is a proper vector, and $(\bar{\mathbf{v}} \cdot \mathbf{S}\bar{\mathbf{v}})$ a proper number, of $\mathbf{S}$. Hence, cases (b) and (c) occur only when $\mathcal{U}_\sigma$ contains proper vectors of $\mathbf{S}$, i.e. when $\sigma \in \{\sigma_1, \sigma_2, \sigma_3\}$, and yield the value zero as a possible local extremum of τ_σ. On the other hand, case (a) corresponds to the typical situation when $\bar{\mathbf{v}}$ is not a proper vector of $\mathbf{S}$.

Next, in order to exploit (a), we put the partial derivatives of ϕ_σ with respect to $\mathbf{v}$, λ, and μ equal to zero and, in addition to the constraint equations

$$\alpha(\mathbf{v}) = 0\,,\ \beta_\sigma(\mathbf{v}) = 0\,, \tag{2.8}$$

we obtain

$$\mathbf{S}^2\mathbf{v} + \hat{\lambda}\,\mathbf{S}\mathbf{v} + \hat{\mu}\,\mathbf{v} = \mathbf{0}, \tag{2.9}$$

where we have set

$$\begin{aligned} \hat{\lambda} &:= 2(\mathbf{v}\cdot\mathbf{v} - 2)(\mathbf{S}\mathbf{v}\cdot\mathbf{v}) + 2\lambda \parallel \mathbf{S}\mathbf{v} - (\mathbf{v}\cdot\mathbf{S}\mathbf{v})\mathbf{v} \parallel, \\ \hat{\mu} &:= (\mathbf{S}\mathbf{v}\cdot\mathbf{v})^2 + 2\mu \parallel \mathbf{S}\mathbf{v} - (\mathbf{v}\cdot\mathbf{S}\mathbf{v})\mathbf{v} \parallel . \end{aligned} \tag{2.10}$$

Our next step is to take the vector product of (2.9) with $\mathbf{S}\mathbf{v}$ and $\mathbf{v}$, in order. On the one hand, for $\mathbf{S}^*$ the cofactor of $\mathbf{S}$, we have that

$$\mathbf{S}\mathbf{v} \times \mathbf{S}^2\mathbf{v} = \mathbf{S}^*(\mathbf{v} \times \mathbf{S}\mathbf{v}), \tag{2.11}$$

and hence

$$\mathbf{S}^*(\mathbf{v} \times \mathbf{S}\mathbf{v}) = \hat{\mu}\,\mathbf{v} \times \mathbf{S}\mathbf{v}. \tag{2.12}$$

On the other hand, for $\mathbf{V}$ the skew tensor associated with $\mathbf{v}$, $\mathbf{S}^T$ the transpose and $\operatorname{tr}\mathbf{S}$ the trace of $\mathbf{S}$, $(\mathbf{S}^T\mathbf{V} + \mathbf{V}\mathbf{S})$ is the skew tensor associated with $((\operatorname{tr}\mathbf{S})\mathbf{v} - \mathbf{S}\mathbf{v})$, [4] and hence

$$(\mathbf{S} - (\operatorname{tr}\mathbf{S})\mathbf{I})(\mathbf{v} \times \mathbf{S}\mathbf{v}) = \hat{\lambda}\,\mathbf{v} \times \mathbf{S}\mathbf{v}. \tag{2.13}$$

In view of (2.12) and (2.13), $\mathbf{v} \times \mathbf{S}\mathbf{v}$ is a proper vector of both $\mathbf{S}^*$ and $\mathbf{S} - \operatorname{tr}(\mathbf{S})\mathbf{I}$, with proper values $\hat{\mu}$ and $\hat{\lambda}$, respectively. Consequently, $\mathbf{v} \times \mathbf{S}\mathbf{v}$ is a proper vector of $\mathbf{S}$ as well; if for some $i \in \{1, 2, 3\}$ we let σ_i be the corresponding proper value, we have that

$$\hat{\mu} = \sigma_j\sigma_k\,, \quad \hat{\lambda} = -(\sigma_j + \sigma_k) \tag{2.14}$$

for $\{i, j, k\} = \{1, 2, 3\}$. With (2.14), (2.9) becomes

$$\mathbf{S}^2\mathbf{v} - (\sigma_j + \sigma_k)\mathbf{S}\mathbf{v} + \sigma_j\sigma_k\mathbf{v} = \mathbf{0}, \tag{2.15}$$

and it follows from (2.3) and (2.15) that

$$\tau_\sigma(\mathbf{v}) = \sqrt{-\sigma^2 + (\sigma_j + \sigma_k)\sigma - \sigma_j\sigma_k} = \sqrt{-(\sigma - \sigma_j)(\sigma - \sigma_k)}. \tag{2.16}$$

Of course, the indices j and k in (2.16) must be chosen so as to make the argument of the square root nonnegative. As a consequence, by also taking into account cases (b) and (c), we find that only the following values may be local extrema of τ_σ:

$$\begin{aligned} &\sqrt{(\sigma_1 - \sigma)(\sigma - \sigma_3)} \text{ and } \sqrt{(\sigma_2 - \sigma)(\sigma - \sigma_3)}, \text{ for } \sigma \in [\sigma_3, \sigma_2], \\ &\sqrt{(\sigma_1 - \sigma)(\sigma - \sigma_3)} \text{ and } \sqrt{(\sigma_1 - \sigma)(\sigma - \sigma_2)}, \text{ for } \sigma \in [\sigma_2, \sigma_1]; \end{aligned} \tag{2.17}$$

hence, for each $\sigma \in [\sigma_3, \sigma_1]$, the mapping τ_σ has exactly two local extrema: one local maximum and one local minimum, whose values are given by (2.17) (cf. (2.4)).

[4] This fact is a consequence of the following property of the trace: given three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and a second-order tensor $\mathbf{T}$,

$$(\operatorname{tr}\mathbf{T})\,\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = \mathbf{T}\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} + \mathbf{a} \times \mathbf{T}\mathbf{b} \cdot \mathbf{c} + \mathbf{a} \times \mathbf{b} \cdot \mathbf{T}\mathbf{c}.$$

Appendix

A. An Account of Mohr's Proof

For simplicity, we treat the typical case when the proper values of $\mathbf{S}$ are all distinct, and hence none of the geometrical objects involved in the proof degenerate. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the orthonormal basis of $\mathcal{V}$ corresponding to the proper values $\sigma_1 > \sigma_2 > \sigma_3$. The main step of Mohr's proof consists in establishing the following statement, which, together with the fact that $\tau(\mathbf{n})$ is nonnegative, implies that the range of the tension mapping is included in Mohr's arbelos:

(i) if the unit vector $\mathbf{n}$ belongs to a cone of axis $\mathbf{e}_1$ [respectively, $\mathbf{e}_2$; $\mathbf{e}_3$], then the point $(\sigma(\mathbf{n}), \tau(\mathbf{n}))$ belongs to a circle, whose centre is located on the $\sigma-$axis at abscissa $(\sigma_2 + \sigma_3)/2$ [respectively, $(\sigma_1 + \sigma_3)/2$; $(\sigma_1 + \sigma_2)/2$], and whose radius is greater than or equal to $(\sigma_2 - \sigma_3)/2$ [respectively, less than or equal to $(\sigma_1 - \sigma_3)/2$; greater than or equal to $(\sigma_1 - \sigma_2)/2$].

Let $\mathbf{e} \in \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $c \in [-1, +1]$ be fixed, and consider the cone $\mathcal{C}(\mathbf{e}, c) := \{\mathbf{v} \in \mathcal{V} \,|\, \mathbf{v} \cdot \mathbf{e} = \| \mathbf{v} \| c\}$. For (σ, τ) to be the image under the tension mapping $\mathcal{T}$ of a unit vector $\mathbf{n} \in \mathcal{C}(\mathbf{e}, c)$, the following system in the unknown $\mathbf{n} \in \mathcal{U}$ must have a solution:

$$\begin{cases} \mathbf{e} \cdot \mathbf{n} = c, \\ \mathbf{S}\mathbf{n} \cdot \mathbf{n} = \sigma, \\ \| \mathbf{S}\mathbf{n} - (\mathbf{S}\mathbf{n} \cdot \mathbf{n})\mathbf{n} \| = \tau. \end{cases} \tag{A.1}$$

The first equation of system (A.1) yields the representation formula

$$\mathbf{n} = \mathbf{m} + c\,\mathbf{e}, \quad \mathbf{m} \in \mathcal{M} := \{\mathbf{v} \in \mathcal{V} \,|\, \mathbf{v} \cdot \mathbf{e} = 0\}. \tag{A.2}$$

Hence, (A.1) may be written as a system for the unknown $\mathbf{m} \in \mathcal{M}$:

$$\begin{cases} \mathbf{m} \cdot \mathbf{m} = 1 - c^2, \\ \mathbf{S}\mathbf{m} \cdot \mathbf{m} = \sigma - c^2 \sigma(\mathbf{e}), \\ \mathbf{S}^2\mathbf{m} \cdot \mathbf{m} = \sigma^2 + \tau^2 - c^2 \sigma(\mathbf{e})^2, \end{cases} \tag{A.3}$$

where $\sigma(\mathbf{e}) := \mathbf{e} \cdot \mathbf{S}\mathbf{e}$, and where the fact that both $\mathbf{S}\mathbf{e}$ and $\mathbf{S}^2\mathbf{e}$ are parallel to $\mathbf{e}$ has been used.

Since $\mathcal{M}$ is an invariant space of $\mathbf{S}$, there is an orthonormal basis of $\mathcal{M}$ made of proper vectors of $\mathbf{S}$, say, $\{\mathbf{f}, \mathbf{g}\}$; moreover, the set $\{\sigma(\mathbf{e}), \sigma(\mathbf{f}), \sigma(\mathbf{g})\}$ is just the set $\{\sigma_1, \sigma_2, \sigma_3\}$. Let $\mathbf{S}_{\mathcal{M}}$ be the restriction of $\mathbf{S}$ to $\mathcal{M}$, and let $\mathbf{I}_{\mathcal{M}}$ be the identity of $\mathcal{M}$. By the Cayley-Hamilton theorem, we have:

$$\mathbf{S}^2_{\mathcal{M}} - [\sigma(\mathbf{f}) + \sigma(\mathbf{g})]\mathbf{S}_{\mathcal{M}} + \sigma(\mathbf{f})\sigma(\mathbf{g})\mathbf{I}_{\mathcal{M}} = \mathbf{0}. \tag{A.4}$$

Multiplying (A.4) by $\mathbf{m} \otimes \mathbf{m}$ and using (A.3), we get

$$\sigma^2 + \tau^2 - c^2\sigma(\mathbf{e})^2 - [\sigma(\mathbf{f}) + \sigma(\mathbf{g})]\,[\sigma - c^2\sigma(\mathbf{e})] + \sigma(\mathbf{f})\sigma(\mathbf{g})(1 - c^2) = 0, \tag{A.5}$$

or, upon reordering,

$$\begin{aligned} &\sigma^2 + \tau^2 - [\sigma(\mathbf{f}) + \sigma(\mathbf{g})]\sigma + \sigma(\mathbf{f})\sigma(\mathbf{g}) + \\ &\quad -c^2\{\sigma(\mathbf{f})\sigma(\mathbf{g}) - \sigma(\mathbf{e})[\sigma(\mathbf{f}) + \sigma(\mathbf{g})] + \sigma(\mathbf{e})^2\} = 0. \end{aligned} \tag{A.6}$$

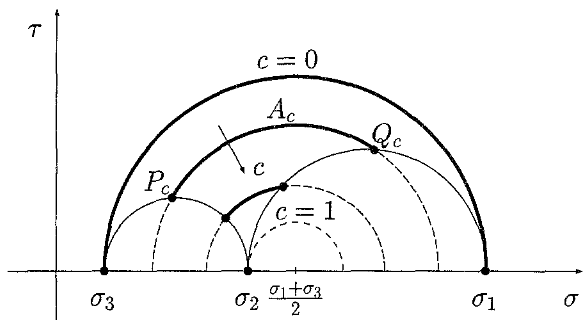

Fig. 3. The arcs A_c as c varies in [0, 1].

This is the equation of a circle, having centre on the σ-axis at abscissa

$$\sigma_o(\mathbf{e}) = \frac{\sigma(\mathbf{f}) + \sigma(\mathbf{g})}{2}, \tag{A.7}$$

and having radius

$$\rho_o(\mathbf{e}, c) := \sqrt{\left(\frac{\sigma(\mathbf{f}) - \sigma(\mathbf{g})}{2}\right)^2 + c^2[\sigma(\mathbf{f}) - \sigma(\mathbf{e})]\,[\sigma(\mathbf{g}) - \sigma(\mathbf{e})]}\,. \tag{A.8}$$

We observe that the coefficient of c^2 in equation (A.8) is nonpositive if $\sigma(\mathbf{e})$ is the intermediate proper value of $\mathbf{S}$, nonnegative otherwise. It follows that $\rho_o(\mathbf{e}, c) \leq \rho_o(\mathbf{e}, 0) = |\sigma(\mathbf{f}) - \sigma(\mathbf{g})|/2$ if $\sigma(\mathbf{e})$ is the intermediate proper value of $\mathbf{S}$, whereas $\rho_o(\mathbf{e}, c) \geq \rho_o(\mathbf{e}, 0)$ otherwise. Since (A.6) may be regarded as a compatibility condition of system (A.1), this completes the proof of statement (*i*).

In particular, when $c = 0$, i.e. when $\mathbf{n}$ is a unit vector orthogonal to $\mathbf{e}$, the point (σ, τ) turns out to belong to one of Mohr's three circles, the one having centre $\sigma_o(\mathbf{e})$ and radius $\rho_o(\mathbf{e}, 0)$. This fact can be used to show that the range of the tension mapping is the whole of Mohr's arbelos.

Indeed, let us consider the family of cones $\{\mathcal{C}(\mathbf{e}_2, c) \mid c \in [0, 1]\}$. For each fixed $c \in [0, 1]$, the image under the tension mapping of the connected set of all the unit vectors of $\mathcal{C}(\mathbf{e}_2, c)$ is a connected subset – in fact, an arc A_c – of a circle $\mathcal{O}_c$ centred at point $(\sigma_3 + \sigma_1)/2$ of the σ-axis, with the radius specified by (A.8) in such a way as to satisfy

$$|\sigma_2 - (\sigma_1 + \sigma_3)/2| \leq \rho_o(\mathbf{e}_2, c) \leq (\sigma_1 - \sigma_3)/2. \tag{A.9}$$

The arc A_c has a point in common with eachone of the two inner Mohr's circles: these points, say P_c and Q_c (Fig. 3), are the images under the tension mapping of the unit vectors of $\mathcal{C}(\mathbf{e}_2, c)$

$$\mathbf{a}_c := c\,\mathbf{e}_2 + \sqrt{1 - c^2}\,\mathbf{e}_3 \quad \text{and} \quad \mathbf{b}_c := c\,\mathbf{e}_2 + \sqrt{1 - c^2}\,\mathbf{e}_1, \tag{A.10}$$

orthogonal to $\mathbf{e}_1$ and $\mathbf{e}_3$, respectively. As a consequence, A_c is precisely the intersection of the circle $\mathcal{O}_c$ with the arbelos. As the parameter c ranges from 0 to 1, the radius of $\mathcal{O}_c$ attains all the values in the interval (A.9), and hence the arc A_c sweeps the whole of Mohr's arbelos.

B. A Topological Approach

Once again, for conciseness, we treat only the case when $\mathbf{S}$ has distinct proper values. We choose a cartesian reference $(O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ for the three-dimensional euclidean point space $\mathcal{E}$, with $O \in \mathcal{E}$ and $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ the proper vectors of $\mathbf{S}$. We then identify a vector $\mathbf{v}$ of $\mathcal{V}$, the translation space of $\mathcal{E}$, with the point $P = O + \mathbf{v}$ of $\mathcal{E}$; with this identification, the tension mapping $\mathcal{T}$ maps S^2, the unit sphere of $\mathcal{E}$, into $\mathbb{R} \times \mathbb{R}^+$.

It is a simple matter to verify that a point $P \in S^2$ of coordinates (x_1, x_2, x_3) has the same image under $\mathcal{T}$ of the point $\overline{P}(|x_1|, |x_2|, |x_3|)$. Therefore, in order to describe the range of $\mathcal{T}$, we can consider the restriction $\overline{\mathcal{T}}$ of $\mathcal{T}$ to the set

$$\overline{S^2} := \{P(x_1, x_2, x_3) \in S^2 \,|\, x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0\}. \tag{B.1}$$

The map $\overline{\mathcal{T}}$ is one-to-one. Indeed, for $P(x_1, x_2, x_3)$, $Q(y_1, y_2, y_3) \in \overline{S^2}$, let $z_i := x_i^2 - y_i^2$ $(i \in \{1, 2, 3\})$; if $\overline{\mathcal{T}}(P) = \overline{\mathcal{T}}(Q)$, then

$$\begin{cases} z_1 + z_2 + z_3 = 0 \\ \sigma_1 z_1 + \sigma_2 z_2 + \sigma_3 z_3 = 0 \\ \sigma_1^2 z_1 + \sigma_2^2 z_2 + \sigma_3^2 z_3 = 0, \end{cases} \tag{B.2}$$

a Vandermonde system whose unique solution is $z_1 = z_2 = z_3 = 0$; hence $P \equiv Q$.

The sets $\overline{S^2}$ and $\mathcal{T}(\mathcal{U})$ inherit by restriction the euclidean topology of $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively; in particular, $\overline{S^2}$ is compact, and $\mathcal{T}(\mathcal{U})$ is Hausdorff. As a consequence, the one-to-one continuous map $\overline{\mathcal{T}}$ is a homeomorphism. Thus, since $\overline{S^2}$ is a (two-dimensional, compact) topological manifold with boundary, such is $\mathcal{T}(\mathcal{U})$, and the boundary of the latter is the image of the boundary of the former under $\overline{\mathcal{T}}$. But, as an easy computation shows, eachone of the three quarter circles forming the boundary of $\overline{S^2}$ is mapped by $\overline{\mathcal{T}}$ onto one of the three half circles that all together bound Mohr's arbelos.

Acknowledgements

This research was supported by M.U.R.S.T., Progetto Nazionale 'Termomeccanica dei Continui Classici e dei Materiali Nuovi'.

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