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Geometry

We identify the shell with a two dimensional surface $\mathcal{S}$, which we use as *label set* to identify the particles that comprise the shell. The label set should not be confused with the *configuration* of the shell, the latter being specified by

- a *placement* $\boldsymbol{y} : \mathcal{S} \to \mathcal{E}$

and

- a *director field* $\boldsymbol{d} : \mathcal{S} \to \text{Unit sphere} = \{\boldsymbol{v} \in T\mathcal{E} : |\boldsymbol{v}| = 1\}.$

Equilibrium

We postulate [1] the existence of pair of tensor fields $\boldsymbol{S}$ and $\boldsymbol{M}$ such that the resultant force and the resultant moment exerted on a part $\mathcal{P} \subset \mathcal{S}$ are given by

$$\boldsymbol{r}(\mathcal{P}) = \int_{x\in\mathcal{P}} \boldsymbol{b} + \int_{x\in\partial\mathcal{P}} \boldsymbol{S}\boldsymbol{\nu}, \qquad \boldsymbol{m}(\mathcal{P}) = \int_{x\in\mathcal{P}} (\boldsymbol{y}\times\boldsymbol{b} + \boldsymbol{d}\times\boldsymbol{c}) + \int_{x\in\partial\mathcal{P}} (\boldsymbol{y}\times\boldsymbol{S}\boldsymbol{\nu} + \boldsymbol{d}\times\boldsymbol{M}\boldsymbol{\nu}). \quad (1)$$

In the above expressions, $\boldsymbol{b}(x)$ and $\boldsymbol{d}(x) \times \boldsymbol{c}(x)$ are, respectively, the applied force and the applied couple per unit *reference area* at x. In the boundary integrals, $\boldsymbol{\nu}(x) \in T_x\mathcal{S}$ denotes the unit vector tangent to $\mathcal{S}$, orthogonal to the curve $\partial\mathcal{P}$, and pointing outside $\mathcal{P}$ at x. The terms $\boldsymbol{S}(x)\boldsymbol{\nu}(x)$ and $\boldsymbol{d}(x) \times \boldsymbol{M}(x)\boldsymbol{\nu}(x)$ represent, respectively, the linear densities of *contact force* and *contact couple* applied at the boundary of $\mathcal{P}$ at x.

Remarks:

- The fact that surface and line couples are orthogonal to $\boldsymbol{d}$ encodes the assumption that a shell cannot sustain and transmit couples parallel to the director.
- We take both $\boldsymbol{c}$ and $\boldsymbol{M}\boldsymbol{\nu}$ orthogonal to $\boldsymbol{d}$.

- While the tensor $\boldsymbol{S}(x)$ is a linear map from $T_x\mathcal{S}$ to $T\mathcal{E}$, $\boldsymbol{M}(x)$ is a linear map from $T_x\mathcal{S}$ to the orthogonal complement of the space spanned by $\boldsymbol{d}(x)$:

$$\boldsymbol{M}(x): T_x\mathcal{S} \to \operatorname{span}\{\boldsymbol{d}(x)\}^\perp, \qquad \operatorname{span}\{\boldsymbol{d}\}^\perp = \{\boldsymbol{v} \in T\mathcal{E} : \boldsymbol{v}\cdot\boldsymbol{d} = 0\}. \tag{2}$$

- The tensor fields $\boldsymbol{S}$ and $\boldsymbol{M}$ are the homologous of the Piola (AKA 1st Piola-Kirchhoff) stress.
- As long as we are concerned with balance equations, it does not matter whether the surface $\mathcal{S}$ is flat or curved. In fact, the differential equations of equilibrium we write are independent on the choice of $\mathcal{S}$. We may take $\mathcal{S}$ to be any two-dimensional manifold (even flat). However, taking $\mathcal{S}$ to be a surface makes the resulting equations more compact when we develop a small-displacement, small-strain theory.
- All tensor fields we shall consider are superficial: their domain at x is $T_x\mathcal{S}$.
- We shall call a tensor field *tangential* if its range is $T_x\mathcal{S}$.

Divergence theorems

We shall use tangential vector and tensor fields to represent scalar and vector-valued *fluxes* on $\mathcal{S}$. Accordingly, we shall need divergence theorems to transform line integrals over the boundary $\partial\mathcal{P}$ of a part $\mathcal{P}\subset\mathcal{S}$ into surface integrals over $\mathcal{P}$. To begin with, we record that if $\boldsymbol{v}$ is a vector field on $\mathcal{S}$ (be it tangential or not), then the *gradient* of $\boldsymbol{v}$ may be defined using a coordinate system on $\mathcal{S}$ by setting

$$\nabla\boldsymbol{v} = \boldsymbol{v}_{,\alpha}\otimes\mathbf{a}^\alpha, \tag{3}$$

with $\mathbf{a}^\alpha$ the elements of the contravariant basis of $\mathcal{S}$, the dual of the covariant basis $\mathbf{a}_\alpha = x_{,\alpha}$, and $\mathbf{P}(x) = \boldsymbol{I} - \boldsymbol{n}(x)\otimes\boldsymbol{n}(x)$ the projector on the tangent space $T_x\mathcal{S}$. More intrinsic notions of gradient, which seemingly do not make use of coordinates may be used, however, since the very notion of manifold is based on the introduction of charts and atlases, we do not see any reason for avoiding coordinates in the present situation. As a matter of fact, a more radical approach would rule out the metric of $\mathcal{S}$ and work with differential forms (as far as balance laws are concerned).

If $\mathbf{v}$ is a *tangential* vector field, then the following divergence theorem holds:

$$\int_{\partial\mathcal{P}}\mathbf{v}\cdot\boldsymbol{\nu} = \int_{\mathcal{P}}\operatorname{div}\mathbf{v}, \quad \text{where} \quad \operatorname{div}\mathbf{v} = \operatorname{tr}\nabla\mathbf{v} = \mathbf{v}_{,\alpha}\cdot\mathbf{a}^\alpha. \tag{4}$$

Next, consider the tensor field $\boldsymbol{B} = \boldsymbol{b}\otimes\mathbf{v}$ where $\boldsymbol{b}$ is a constant vector and $\mathbf{v}$ is a *tangential tensor field*. Then

$$\int_{\mathcal{P}}\boldsymbol{B}\boldsymbol{\nu} = \boldsymbol{b}\otimes\int_{\partial\mathcal{P}}\boldsymbol{v}\cdot\boldsymbol{\nu} = \boldsymbol{b}\otimes\int_{\mathcal{P}}\operatorname{div}\boldsymbol{v} = \int_{\mathcal{P}}\boldsymbol{B}_{,\alpha}\mathbf{a}^\alpha. \tag{5}$$

Let $\{\boldsymbol{e}_i, i=1,2,3\}$ be a basis for the space of translations of the physical space $\mathcal{E}$. Since every tensor field $\boldsymbol{T}$ can be written as $\boldsymbol{T} = \boldsymbol{e}_i\otimes\boldsymbol{v}^{(i)}$, for a suitable choice of a triplet of vector fields $\boldsymbol{v}^{(i)}$, the identity (5) holds for whatever tensor field $\boldsymbol{T}$. Thus, we can conveniently take

$$\operatorname{div}\boldsymbol{T} = \boldsymbol{T}_{,\alpha}\mathbf{a}^\alpha = \nabla\boldsymbol{T}(\cdot,\mathbf{a}^\alpha) = \nabla\boldsymbol{T} : \mathbf{P}. \tag{6}$$

as the definition of divergence of a tensor field $\boldsymbol{T}$. Here double dots denote contraction with respect to the last two indices.

Pointwise form of the equilibrium equations

We require that the resultant force $\boldsymbol{r}(\mathcal{P})$ and the resultant moment $\boldsymbol{m}(\mathcal{P})$ be null for every part $\mathcal{P}\subset\mathcal{S}$. It is not difficult, using the divergence theorem and a standard localization argument to deduce the equivalence:

$$\boldsymbol{r}(\mathcal{P})=\mathbf{0}\quad\forall\mathcal{P}\subset\mathcal{S}$$
$$\Updownarrow$$
$$\boxed{\operatorname{div}\boldsymbol{S}+\boldsymbol{b}=\mathbf{0}\quad\text{everywhere in }\mathcal{S}.}$$

A similar argument can be applied to the equilibrium of moments. However, the result of this procedure is more transparent if we use the identity

$$(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}=(\boldsymbol{a}\otimes\boldsymbol{b}-\boldsymbol{b}\otimes\boldsymbol{a})\boldsymbol{c}=2\,\mathrm{skw}(\boldsymbol{a}\otimes\boldsymbol{b})\boldsymbol{c}.\tag{7}$$

Then the the vanishing of the resultant moment for each part $\mathcal{P}$ is equivalent to

$$\mathbf{0}=\mathrm{skw}\Big[\int_{\mathcal{P}}(\boldsymbol{y}\otimes\boldsymbol{b}+\boldsymbol{d}\otimes\boldsymbol{c})+\int_{\partial\mathcal{P}}(\boldsymbol{y}\otimes\boldsymbol{S}\boldsymbol{\nu}+\boldsymbol{d}\otimes\boldsymbol{M}\boldsymbol{\nu})\Big]\qquad\forall\mathcal{P}\subset\mathcal{S},\tag{8}$$

Then same procedure as before yields the following equivalence

$$\boldsymbol{m}(\mathcal{P})=\mathbf{0}\quad\forall\mathcal{P}\subset\mathcal{S}$$
$$\Updownarrow$$
$$\mathrm{skw}\big(\boldsymbol{d}\otimes(\operatorname{div}\boldsymbol{M}+\boldsymbol{c})+\boldsymbol{y}\otimes(\operatorname{div}\boldsymbol{S}+\boldsymbol{b})+\nabla\boldsymbol{y}\,\boldsymbol{S}^T+\nabla\boldsymbol{d}\,\boldsymbol{M}^T\big)=\mathbf{0}\quad\text{in }\mathcal{S},$$

On introducing the *deformation gradient* and the *director gradient*

$$\boxed{\boldsymbol{F}=\nabla\boldsymbol{y},\qquad\boldsymbol{G}=\nabla\boldsymbol{d},}\tag{9}$$

on taking into account the equilibrium equation $\operatorname{div}\boldsymbol{S}+\boldsymbol{b}=\mathbf{0}$, and on recalling that $\mathrm{skw}\,\boldsymbol{A}=-(\mathrm{skw}\,\boldsymbol{A})^T=-\,\mathrm{skw}(\boldsymbol{A}^T)$ for every tensor $\boldsymbol{A}$, the last equation yields

$$\mathrm{skw}\big(\boldsymbol{d}\otimes(\operatorname{div}\boldsymbol{M}+\boldsymbol{c})-\boldsymbol{S}\boldsymbol{F}^T-\boldsymbol{M}\boldsymbol{G}^T\big)=\mathbf{0}\quad\text{in }\mathcal{S}\tag{10}$$

Note that this equation coincides with the symmetry condition (5.18) found in [^1]

We now define

$$\boldsymbol{P}=\boldsymbol{I}-\boldsymbol{d}\otimes\boldsymbol{d}.\tag{11}$$

Warning: $\boldsymbol{P}$ should not be confused with $\mathbf{P}=\boldsymbol{I}-\boldsymbol{n}\otimes\boldsymbol{n}$. We try to stick to the following rule: slanted boldface symbols denote vectors in $T\mathcal{E}$ or tensor fields having their range in $T\mathcal{E}$. Upright boldface symbol denote tangential vector fields or tangential tensor fields.

Thanks to the above decomposition we now can write:

$$\boldsymbol{S}=(\boldsymbol{d}\otimes\boldsymbol{d})\boldsymbol{S}+\boldsymbol{P}\boldsymbol{S}=\boldsymbol{d}\otimes\boldsymbol{S}^T\boldsymbol{d}+\boldsymbol{P}\boldsymbol{S},\tag{12}$$

Consequently, (10) can be written as

$$\mathrm{skw}\big(\boldsymbol{d}\otimes(\operatorname{div}\boldsymbol{M}+\boldsymbol{c})-\underbrace{(\boldsymbol{d}\otimes\boldsymbol{d})\boldsymbol{S}\boldsymbol{F}^T}_{\boldsymbol{d}\otimes\boldsymbol{F}\boldsymbol{S}^T\boldsymbol{d}}-\boldsymbol{P}\boldsymbol{S}\boldsymbol{F}^T-\boldsymbol{M}\boldsymbol{G}^T\big)=\mathbf{0}\quad\text{in }\mathcal{S}\tag{13}$$

Whence,

$$\mathrm{skw}\,\boldsymbol{A}=\mathbf{0},\qquad\text{where}\quad\boldsymbol{A}=\boldsymbol{d}\otimes(\operatorname{div}\boldsymbol{M}+\boldsymbol{c}-\boldsymbol{F}\boldsymbol{S}^T\boldsymbol{d})-\boldsymbol{P}\boldsymbol{S}\boldsymbol{F}^T-\boldsymbol{M}\boldsymbol{G}^T\tag{14}$$

A consequence of the last equation is that $\boldsymbol{P}(\mathrm{skw}\,\boldsymbol{A})\boldsymbol{d}=\mathbf{0}$, whence (recall that $\boldsymbol{P}\boldsymbol{c}=\mathbf{0}$):

$$\boxed{\boldsymbol{P}(\operatorname{div}\boldsymbol{M}-(\boldsymbol{F}\,\boldsymbol{S}^T+\boldsymbol{S}\boldsymbol{F}^T)\boldsymbol{d})+\boldsymbol{c}=\mathbf{0}.}\tag{15}$$

Another consequence is obtained by writing $\boldsymbol{P}(\operatorname{skw}\boldsymbol{A})\boldsymbol{P} = \mathbf{0}$, which gives:

$$\boxed{\operatorname{skw}(\boldsymbol{PSF}^T + \boldsymbol{MG}^T) = \mathbf{0}.} \tag{16}$$

The aforementioned calculation is based on the splitting

$$\boldsymbol{S} = \boldsymbol{N} + \boldsymbol{d} \otimes \mathbf{q} \tag{17}$$

where

$$\boldsymbol{N} = \boldsymbol{PS}, \qquad \mathbf{q} = \boldsymbol{F}^T\boldsymbol{d}. \tag{18}$$

The equilibrium equation for the forces becomes

$$\operatorname{div}\boldsymbol{N} + \boldsymbol{G}\boldsymbol{q} + (\operatorname{div}\mathbf{q})\boldsymbol{d} + \boldsymbol{b} = \mathbf{0} \tag{19}$$

The above equilibrium equations can be written as

$$\boxed{\boldsymbol{P}(\operatorname{div}\boldsymbol{M} - \boldsymbol{F}\mathbf{q}) + \boldsymbol{NF}^T\boldsymbol{d} + \boldsymbol{c} = \mathbf{0},} \tag{20}$$

$$\boxed{\operatorname{skw}(\boldsymbol{NF}^T + \boldsymbol{MG}^T) = \mathbf{0}.} \tag{21}$$

- The vector $\mathbf{q} = \boldsymbol{S}^T\boldsymbol{d}$ is the *shear vector*. Its interpretation is as follows: the scalar product $\mathbf{q}\cdot\boldsymbol{\nu}$ represents the component along $\boldsymbol{d}$ of the line contact force acting across a line element from the side of $\boldsymbol{\nu}$.
- The tensor $\boldsymbol{N}$ is the membrane-force tensor. Note that in a theory of large displacements $\boldsymbol{N}$ is *not a tangential field*: its range is orthogonal to $\boldsymbol{d}$, not to $\boldsymbol{n}$ (the referential unit normal).

Equilibrium in the reference configuration and component representation

Consider the special case when $\boldsymbol{y}(\boldsymbol{x}) = \boldsymbol{x}$ and $\boldsymbol{d} = \boldsymbol{n}$. Then, the equilibrium equations become

$$\boxed{\mathbf{P}(\operatorname{div}\mathbf{N} + \boldsymbol{b}) + \mathbf{G}\mathbf{q} = \mathbf{0}} \tag{22}$$

$$\boxed{\operatorname{div}\mathbf{q} - \mathbf{N}\cdot\mathbf{G} + \boldsymbol{b}\cdot\boldsymbol{n} = 0} \tag{23}$$

$$\boxed{\mathbf{P}\operatorname{div}\mathbf{M} - \mathbf{q} + \boldsymbol{c} = \mathbf{0},} \tag{24}$$

$$\boxed{\operatorname{skw}(\mathbf{N} + \mathbf{M}\mathbf{G}^T) = \mathbf{0}.} \tag{25}$$

Here we use upright boldface fonts to denote tensors fields whose value at a point x is a map $T_x\mathcal{S} \to T_x\mathcal{S}$, i.e., tangential tensor fields. Thus, in what follows $\mathbf{P}(\boldsymbol{x})$ will denote the projector on the tangent space at $\boldsymbol{x}$.

Most references dealing with thin shells contain equilibrium equations written using contravariant components. To compare our result with theirs, we introduce the representation

$$\mathbf{N} = N^{\alpha\beta}\mathbf{a}_\alpha \otimes \mathbf{a}_\beta, \qquad \mathbf{q} = q^\alpha\mathbf{a}_\alpha, \qquad \boldsymbol{M} = M^{\alpha\beta}\mathbf{a}_\alpha \otimes \mathbf{a}_\beta, \tag{26}$$

where $\mathbf{a}_\alpha = x_{,\alpha}$ is the covariant basis. Furthermore, we write

$$\mathbf{G} = \nabla\boldsymbol{n} = \boldsymbol{n}_{,\alpha} \otimes \mathbf{a}^\alpha = (\boldsymbol{n}_{,\beta}\cdot\mathbf{a}^\alpha)\mathbf{a}_\alpha \otimes \mathbf{a}^\beta = -B^\alpha_{.\beta}\mathbf{a}_\alpha \otimes \mathbf{a}^\beta = -B_{\alpha\beta}\mathbf{a}^\alpha \otimes \mathbf{a}^\beta. \tag{27}$$

We then have

$$\boldsymbol{P}\operatorname{div}\mathbf{N} = \boldsymbol{N}_{,\beta}\mathbf{a}^\beta = N^{\alpha\beta}|_\beta\mathbf{a}_\alpha, \qquad \mathbf{G}\mathbf{q} = -B^\alpha_{.\beta}q^\beta\mathbf{a}_\alpha, \qquad \mathbf{P}\boldsymbol{b} = b^\alpha\mathbf{a}_\alpha. \tag{28}$$

Thus, the first equilibrium equation becomes

$$\boxed{N^{\alpha\beta}|_{\beta} - B^{\alpha}_{.\beta} q^{\beta} + b^{\alpha}, \qquad \alpha = 1, 2.} \tag{29}$$

Furthermore,

$$\operatorname{div} \boldsymbol{q} = \boldsymbol{q}_{,\alpha} \cdot \boldsymbol{a}^{\alpha} = q^{\alpha}|_{\alpha}, \qquad \mathbf{N} \cdot \mathbf{G} = -N^{\alpha\beta} B_{\alpha\beta} \tag{30}$$

Hence the second equilibrium equation becomes

$$\boxed{q^{\alpha}|_{\beta} + B_{\alpha\beta} N^{\alpha\beta} + b^{3} = 0.} \tag{31}$$

Analogous calculations performed on the equilibrium equations for the moments yield standard expressions.

Virtual work

We define the external work as:

$$\delta W = \int_{\mathcal{P}} \boldsymbol{b} \cdot \delta \boldsymbol{y} + \boldsymbol{c} \cdot \delta \boldsymbol{d} + \int_{\partial \mathcal{P}} \boldsymbol{S\nu} \cdot \delta \boldsymbol{y} + \boldsymbol{M\nu} \cdot \delta \boldsymbol{d}. \tag{32}$$

On using the divergence theorem

$$\delta W = \int_{\mathcal{P}} \underbrace{(\operatorname{div} \boldsymbol{S} + \boldsymbol{b})}_{0} \cdot \delta \boldsymbol{y} + \underbrace{(\boldsymbol{P} \operatorname{div} \boldsymbol{M} - \boldsymbol{F}\,\boldsymbol{q} + \boldsymbol{c})}_{\mathbf{0}} \cdot \delta \boldsymbol{n} + \underbrace{\boldsymbol{F}\,\boldsymbol{q} \cdot \delta \boldsymbol{d}}_{\boldsymbol{q} \cdot \boldsymbol{F}^{T} \delta \boldsymbol{d}} + \boldsymbol{S} \cdot \delta \boldsymbol{F} + \boldsymbol{M} \cdot \delta \boldsymbol{G} \tag{33}$$

We next write

$$\boldsymbol{S} \cdot \delta \boldsymbol{F} = \boldsymbol{N} \cdot \boldsymbol{F} + \boldsymbol{d} \otimes \boldsymbol{q} \cdot \delta \boldsymbol{F} = \boldsymbol{N} \cdot \delta \boldsymbol{F} + \boldsymbol{q} \cdot \delta \boldsymbol{F}^{T} \boldsymbol{d}. \tag{34}$$

We define the *shear strain*:

$$\boldsymbol{g} = \boldsymbol{F}^{T} \boldsymbol{d}. \tag{35}$$

We arrive at

$$\delta W = \int_{\mathcal{P}} \boldsymbol{N} \cdot \delta \boldsymbol{F} + \boldsymbol{q} \cdot \delta \boldsymbol{g} + \boldsymbol{M} \cdot \delta \boldsymbol{G}. \tag{36}$$

It is easy to see that $\boldsymbol{g}$ is left unaltered by a change of observer, i.e., a transformation of the form $\boldsymbol{F} \mapsto \mathbf{Q}\boldsymbol{F}$, $\boldsymbol{d} \mapsto \mathbf{Q}\boldsymbol{d}$, where $\mathbf{Q} : T\mathcal{E} \to T\mathcal{E}$ is a an orthogonal transformation. Hence $\boldsymbol{g}$ is a strain measure dual to the shear.

We would like to express virtual work in terms of variations of the following strain measures:

$$\mathbf{C} = \boldsymbol{F}^{\top} \boldsymbol{F}, \qquad \mathbf{K} = \boldsymbol{F}^{\top} \boldsymbol{G}. \tag{37}$$

Note that the tensors $\mathbf{C}$ and $\mathbf{K}$ are tangential, in the sense that their range is $T_x\mathcal{S}$. Note in particular that they map $T_x\mathcal{S}$ into itself.

We can establish a connection with the standard strain measures by observing that, on denoting by $\mathbf{a}_{\alpha} = \boldsymbol{x}_{,\alpha}$ and by $\boldsymbol{a}_{\alpha} = \boldsymbol{y}_{,\alpha}$, (again, keep in mind the difference between upright and slanted fonts!) respectively, the covariant basis in the reference and deformed configuration, then

$$\begin{aligned}
\mathbf{C} = \boldsymbol{F}^{T} \boldsymbol{F} &= (\mathbf{a}^{\alpha} \otimes \boldsymbol{a}_{\alpha})(\boldsymbol{a}_{\beta} \otimes \mathbf{a}^{\beta}) = (\boldsymbol{a}_{\alpha} \cdot \boldsymbol{a}_{\beta}) \mathbf{a}^{\alpha} \otimes \mathbf{a}^{\beta} \\
&= a_{\alpha\beta} \mathbf{a}^{\alpha} \otimes \mathbf{a}^{\beta} \\
\mathbf{K} = \boldsymbol{F}^{\top} \boldsymbol{G} &= (\mathbf{a}^{\alpha} \otimes \boldsymbol{a}_{\alpha})(\boldsymbol{d}_{,\beta} \otimes \mathbf{a}^{\beta}) = \boldsymbol{a}_{\alpha} \cdot \boldsymbol{d}_{,\beta} \mathbf{a}^{\alpha} \otimes \mathbf{a}^{\beta} \\
&= \kappa_{\alpha\beta} \mathbf{a}^{\alpha} \otimes \mathbf{a}^{\beta}
\end{aligned}$$

Note also that if we regard $\boldsymbol{F} = \boldsymbol{a}_\alpha \otimes \mathbf{a}^\alpha$ as a map from $T_x\mathcal{S}$ to $T_y\mathcal{S}'$ (here $\mathcal{S}'$ is the image of $\mathcal{S}$), then the inverse $\boldsymbol{F}^{-1}$ is well defined. We now introduce the *referential stress measures*

$$\mathbf{N} = \boldsymbol{F}^{-1}\boldsymbol{N}, \qquad \mathbf{M} = \boldsymbol{F}^{-1}\boldsymbol{M}. \tag{38}$$

Then we can compute

$$\begin{aligned}
\boldsymbol{N}\cdot\delta\boldsymbol{F} + \boldsymbol{M}\cdot\delta\boldsymbol{G} &= \boldsymbol{N}\cdot\dot{\boldsymbol{F}} + \boldsymbol{F}^{-1}\boldsymbol{M}\cdot\boldsymbol{F}^T\delta\boldsymbol{G} \\
&= \boldsymbol{N}\cdot\delta\boldsymbol{F} + \mathbf{M}\cdot\left[\delta(\boldsymbol{F}^T\boldsymbol{G}) - \delta\boldsymbol{F}^T\boldsymbol{G}\right] \\
&= \boldsymbol{N}\cdot\delta\boldsymbol{F} + \mathbf{M}\cdot\delta\mathbf{K} - \boldsymbol{G}\mathbf{M}^T\cdot\delta\boldsymbol{F} \\
&= \left(\boldsymbol{N} - \boldsymbol{G}\mathbf{M}^T\right)\cdot\delta\boldsymbol{F} + \boldsymbol{F}^{-1}\boldsymbol{M}\cdot\delta\boldsymbol{K} \\
&= \left(\boldsymbol{F}^{-1}\boldsymbol{N} - \boldsymbol{F}^{-1}\boldsymbol{G}\mathbf{M}^T\right)\cdot\boldsymbol{F}^\top\delta\boldsymbol{F} + \mathbf{M}\cdot\delta\mathbf{K} \\
&= \left(\mathbf{N} - \widetilde{\mathbf{K}}\mathbf{M}^T\right)\cdot\frac{\delta\mathbf{C}}{2} + \mathbf{M}\cdot\delta\mathbf{K}
\end{aligned}$$

In the last equation we have set $\widetilde{\mathbf{K}} = \boldsymbol{F}^{-1}\boldsymbol{G}$, and have used the fact that $\mathbf{N} - \widetilde{\mathbf{K}}\mathbf{M}^T$ is a symmetric tensor. This follows from (16). In fact, $\boldsymbol{N}\boldsymbol{F}^T + \boldsymbol{M}\boldsymbol{G}^T \in \mathrm{Sym}$ is equivalent to $\boldsymbol{F}^{-1}(\boldsymbol{N}\boldsymbol{F}^T + \boldsymbol{M}\boldsymbol{G}^T)\boldsymbol{F}^{-T} \in \mathrm{Sym}$, that is to say, $\mathbf{N} + \mathbf{M}\widetilde{\mathbf{K}}^T \in \mathrm{Sym}$. But this in turn is equivalent to to $\mathbf{N} - (\mathbf{M}\widetilde{\mathbf{K}}^T)^T \in \mathrm{Sym}$. Note that:

$$\widetilde{\mathbf{K}} = \boldsymbol{F}^{-1}\boldsymbol{G} = \kappa^{\alpha}_{\cdot\beta}\mathbf{a}^\alpha \otimes \mathbf{a}^\beta, \qquad \kappa^{\alpha}_{\cdot\beta} = \boldsymbol{a}^\alpha\cdot\boldsymbol{d}_{,\beta} \tag{39}$$

We introduce the Green-Lagrange strains

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathring{\mathbf{C}}), \qquad \mathbf{L} = \mathbf{K} - \mathring{\mathbf{K}}. \tag{40}$$

where $\mathring{\mathbf{C}} = \nabla\boldsymbol{x} = \mathbf{P}$ and $\mathring{\mathbf{K}} = \nabla\boldsymbol{n}$ are the initial values of $\mathbf{C}$ and $\mathbf{K}$.

Then the above equation becomes

$$\delta W(\mathcal{P}) = \int_{\mathcal{P}} \left(\mathbf{N} - \widetilde{\mathbf{K}}\mathbf{M}^T\right)\cdot\delta\mathbf{E} + \mathbf{q}\cdot\delta\mathbf{g} + \mathbf{M}\cdot\delta\mathbf{L}$$

Remarks

- the book by Taylor, Zienkiewicz and Fox obtains the same expression of the internal power by starting from three-dimensional elasticity. We work out the expression of the stress working and our frame-indifference requirements without invoking a parent three-dimensional theory.
- The stress measures $\mathbf{M}$ and $\mathbf{N}$ are the homologous of the Cosserat (AKA 2nd Piola-Kirchhoff) stress in nonlinear elasticity
- As the notation suggests, $\boldsymbol{C}$ is the analogue of the right Green strain.
- Our derivation is completely component- and derivative-free. In fact, a definition of gradient of scalar and tensor field can be given, based on the notion of tangent space and Gateaux derivative can be given, which does not rely on coordinates (although in differential geometry the very notion of manifold requires the introduction of chart, and hence a coordinate system.)

Unshearable shells

From this point on, we shall restrict our attention to shells satisfying the constraint:

$$\mathbf{g} = \boldsymbol{F}^T\boldsymbol{d} = \mathbf{0}. \tag{41}$$

This constraint corresponds to the requirement that the director be orthogonal to the tangent space to the shell in the current configuration. On taking the gradient of the equation $\boldsymbol{F}^T\boldsymbol{d} = \boldsymbol{F}[\boldsymbol{d},\cdot] = \boldsymbol{0}$, we find the relation $\boldsymbol{F}^T\boldsymbol{G} + \nabla\boldsymbol{F}[\boldsymbol{d},\cdot,\cdot]$. Note that, $\nabla\boldsymbol{F}[\boldsymbol{n},\cdot,\cdot] = 0$, regarded as a tensor, has its range orthogonal to $\boldsymbol{d}$. [2]

As a consequence, the map $\boldsymbol{F} : T_{\boldsymbol{x}}\mathcal{S} \to T_{\boldsymbol{y}(\boldsymbol{x})}\mathcal{S}'$ is invertible, whence

$$\boldsymbol{G} = \boldsymbol{F}^{-T}\nabla\boldsymbol{F}[\boldsymbol{d},\cdot,\cdot]. \tag{42}$$

Observe that in this case we have

$$\boldsymbol{P}\boldsymbol{F} = \boldsymbol{F} - (\boldsymbol{d}\otimes\boldsymbol{d})\boldsymbol{F} = \boldsymbol{F}. \tag{43}$$

Thus the equilibrium equations become

$$\boxed{\boldsymbol{P}\,\mathrm{div}\,\boldsymbol{M} - \boldsymbol{F}\boldsymbol{q} + \boldsymbol{c} = \boldsymbol{0},} \tag{44}$$

$$\boxed{\mathrm{skw}(\boldsymbol{N}\boldsymbol{F}^T + \boldsymbol{M}\boldsymbol{G}^T) = \boldsymbol{0}.} \tag{45}$$

It is possible to rule out the shear from the equations. By making use of the decomposition $\boldsymbol{S} = \boldsymbol{N} + \boldsymbol{n}\otimes\boldsymbol{q}$ with $\boldsymbol{N}^T\boldsymbol{n} = \boldsymbol{0}$, we can write the equilibrium equation for the forces as

$$\mathrm{div}\,\boldsymbol{N} + \boldsymbol{G}\boldsymbol{q} + (\mathrm{div}\,\boldsymbol{q})\boldsymbol{n} + \boldsymbol{b} = \boldsymbol{0}. \tag{46}$$

Note that $\boldsymbol{n}\cdot\mathrm{div}\,\boldsymbol{N} = \mathrm{div}(\boldsymbol{N}^T\boldsymbol{n}) - \boldsymbol{N}\cdot\nabla\boldsymbol{n}$. Note also that the range of $\boldsymbol{N}$ is orthogonal to $\boldsymbol{n}$. Thus, we conclude that $\boldsymbol{n}\cdot\mathrm{div}\,\boldsymbol{N} = -\boldsymbol{N}\cdot\nabla\boldsymbol{n} = -\boldsymbol{N}\cdot\boldsymbol{G}$. We also note that $\boldsymbol{G}^T\boldsymbol{n} = \boldsymbol{0}$. Thus, taking the scalar product of (46) with $\boldsymbol{n}$, we obtain

$$\mathrm{div}\,\boldsymbol{q} = \boldsymbol{N}\cdot\boldsymbol{G} - \boldsymbol{n}\cdot\boldsymbol{b}. \tag{47}$$

Next, we pre-multiply (44) by $\boldsymbol{F}^{-1}$ to obtain (again, note that $\boldsymbol{F}^{-1}$ is well defined as a map from $T_{\boldsymbol{y}(\boldsymbol{x})}\mathcal{S}' \to T_{\boldsymbol{x}}\mathcal{S}$:

$$\boldsymbol{q} = \boldsymbol{F}^{-1}(\boldsymbol{P}\,\mathrm{div}\,\boldsymbol{M} + \boldsymbol{c}). \tag{48}$$

and we take the divergence. Then, we can use the above equation to eliminate $\mathrm{div}\,\boldsymbol{q}$. The result is:

$$\boxed{\mathrm{div}(\boldsymbol{F}^{-1}(\boldsymbol{P}\,\mathrm{div}\,\boldsymbol{M} + \boldsymbol{c})) - \boldsymbol{N}\cdot\boldsymbol{G} - \boldsymbol{n}\cdot\boldsymbol{b} = \boldsymbol{0}.} \tag{49}$$

This equation must be coupled with what remains from (46):

$$\boxed{\boldsymbol{P}(\mathrm{div}\,\boldsymbol{N} + \boldsymbol{b}) + \boldsymbol{G}\boldsymbol{F}^{-1}(\boldsymbol{P}\,\mathrm{div}\,\boldsymbol{M} + \boldsymbol{c}) = \boldsymbol{0}.} \tag{50}$$

Unshearable hyperelastic shells

Assume that the Helmholtz free energy per unit referential area is

$$\psi = \widehat{\psi}(\mathbf{E},\mathbf{L}). \tag{51}$$

The Dissipation Principle demands that, given any part $\mathcal{P}\subset\mathcal{S}$, the time derivative of the free energy be not greater than the external power:

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathcal{P}}\psi \le \int_{\mathcal{P}}\boldsymbol{b}\cdot\dot{\boldsymbol{y}} + \boldsymbol{c}\cdot\dot{\boldsymbol{d}} + \int_{\partial\mathcal{P}}\boldsymbol{S}\boldsymbol{\nu}\cdot\dot{\boldsymbol{y}} + \boldsymbol{M}\boldsymbol{\nu}\cdot\dot{\boldsymbol{d}} \qquad \forall\mathcal{P}\subset\mathcal{S}. \tag{52}$$

By making use of the principle of virtual powers, the above inequality is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{P}}\psi \le \int_{\mathcal{P}}\left(\mathbf{N}-\widetilde{\mathbf{K}}\mathbf{M}^T\right)\cdot\dot{\mathbf{E}}+\mathbf{M}\cdot\dot{\mathbf{L}} \tag{53}$$

A standard argument yields, for a hyperelastic material, the constitutive equations for the Cosserat-like stresses are

$$\mathbf{N}=\partial_{\mathbf{E}}\widehat{\psi}(\mathbf{E},\mathbf{L})+\widetilde{\mathbf{K}}\mathbf{M}^T \qquad \text{and} \qquad \mathbf{M}=\partial_{\mathbf{L}}\widehat{\psi}(\mathbf{E},\mathbf{L}).$$

As a result, the constitutive equation for the Piola-like stresses are

$$\boldsymbol{N}=\boldsymbol{F}\mathbf{N}=\boldsymbol{F}\partial_{\mathbf{E}}\widehat{\psi}\left(\frac{1}{2}(\boldsymbol{F}^T\boldsymbol{F}-\mathbf{P}),\boldsymbol{F}^T\boldsymbol{G}-\mathring{\mathbf{K}}\right)+\boldsymbol{G}\widehat{\boldsymbol{M}}^T(\boldsymbol{F},\boldsymbol{G})\boldsymbol{F}^{-T}=:\widehat{\boldsymbol{N}}(\boldsymbol{F},\boldsymbol{G})$$

$$\boldsymbol{M}=\boldsymbol{F}\mathbf{M}=\boldsymbol{F}\partial_{\mathbf{L}}\widehat{\psi}\left(\frac{1}{2}(\boldsymbol{F}^T\boldsymbol{F}-\mathbf{P}),\boldsymbol{F}^T\boldsymbol{G}-\mathring{\mathbf{K}}\right)=:\widehat{\boldsymbol{M}}(\boldsymbol{F},\boldsymbol{G})$$

Remark. Although $\mathbf{E}$ and $\mathbf{L}$ are in duality with $\mathbf{N}$ and $\mathbf{M}$, they do not determine them constitutively.

Koiter shells

We for h a thickness parameter, we assume that the strain energy is separable, and has the form

$$\psi=\widehat{\psi}(\mathbf{E},\mathbf{L})=\frac{h}{2}\mathbb{C}\left[\mathbf{E}-\mathbf{E}_R\right]\cdot\mathbf{E}+\frac{h^3}{24}\mathbb{C}[\mathbf{L}-\mathbf{L}_R]\cdot(\mathbf{L}-\mathbf{L}_R). \tag{54}$$

where $\mathbf{E}_R$ and $\mathbf{L}_R$ are symmetric tensors, which we interpret as residual straand $\mathbb{C}$ is the fourth-order tensor defined by

$$\mathbb{C}[\mathbf{E}]=2\widetilde{\mu}\,\mathrm{sym}\,\mathbf{E}+\tilde{\lambda}\,\mathrm{tr}(\mathbf{E})\mathring{\mathbf{P}}, \tag{55}$$

where $\widetilde{\mu}$ and $\tilde{\lambda}$ are the *effective Lamé constants*. We compute

$$\mathbf{N}=h\mathbb{C}[\mathbf{E}-\mathbf{E}_R]+\widetilde{\mathbf{K}}\mathbf{M}^T\in\mathrm{Sym}, \qquad \mathbf{M}=\mathbf{M}_0+\frac{h^3}{12}\mathbb{C}[\mathbf{K}]\in\mathrm{Sym}, \tag{56}$$

We write

$$\widetilde{\mathbf{L}}=\widetilde{\mathbf{K}}-\mathring{\widetilde{\mathbf{K}}}=\widetilde{\mathbf{K}}-\mathring{\mathbf{K}}=\boldsymbol{F}^{-1}\boldsymbol{G}-\mathring{\mathbf{K}}. \tag{57}$$

Then the constitutive equations for hyperelastic shells specialize to

$$\begin{aligned}\mathbf{N}&=h\mathbb{C}[\mathbf{E}-\mathbf{E}_R]+\widetilde{\mathbf{K}}\mathbf{M}^T=\mathring{\mathbf{N}}+h\mathbb{C}[\mathbf{E}]+\widetilde{\mathbf{L}}\mathbf{M}+\frac{h^3}{12}\mathring{\mathbf{K}}\mathbb{C}[\mathbf{L}],\\ \mathbf{M}&=\frac{h^3}{12}\mathbb{C}[\mathbf{L}-\mathbf{L}_R]=\mathring{\mathbf{M}}+\frac{h^3}{12}\mathbb{C}[\mathbf{L}],\end{aligned}$$

where $\mathring{\mathbf{M}}=\mathring{\mathbf{M}}^T=-\frac{h^3}{12}\mathbb{C}[\mathbf{L}_R]$, and $\mathring{\mathbf{N}}=-h\mathbb{C}[\mathbf{E}_R]+\mathring{\mathbf{K}}\mathring{\mathbf{M}}$.

The expressions of the Piola-like stresses $\boldsymbol{N}$ and $\boldsymbol{M}$ in terms of their duals $\boldsymbol{F}$ and $\boldsymbol{G}$ are:

$$\boxed{\begin{aligned}&\boldsymbol{N}=\boldsymbol{F}\mathbf{N}=\boldsymbol{F}\mathring{\mathbf{N}}+h\boldsymbol{F}\mathbb{C}\left[\frac{1}{2}(\boldsymbol{F}^T\boldsymbol{F}-\mathbf{I})\right]+(\boldsymbol{G}-\boldsymbol{F}\mathring{\mathbf{K}})\mathring{\mathbf{M}}+\frac{h^3}{12}\boldsymbol{G}\mathbb{C}[\boldsymbol{F}^T\boldsymbol{G}-\mathring{\mathbf{K}}]\\ &\boldsymbol{M}=\boldsymbol{F}\mathring{\mathbf{M}}+\frac{h^3}{12}\boldsymbol{F}\mathbb{C}[\boldsymbol{F}^T\mathbf{G}-\mathring{\mathbf{K}}]\end{aligned}} \tag{58}$$

Small displacements and rotations

We write $\boldsymbol{y}(\boldsymbol{x}) = \boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x})$ and $\boldsymbol{d}(\boldsymbol{x}) = \boldsymbol{n}(\boldsymbol{x}) + \boldsymbol{\varphi}(\boldsymbol{x})$. We refer to $\boldsymbol{u}$ and $\boldsymbol{\varphi}$, respectively, as the *displacement* and the *rotation*. We notice that, since $\boldsymbol{n}$ has norm 1, the rotation is a tangential field, *i.e.*, $\boldsymbol{\varphi} \cdot \boldsymbol{n} = 0$.

We write (note that $\mathbf{P}$ should not be confused with $\boldsymbol{P}$) . In fact, $\mathbf{P}$ is the value of $\boldsymbol{P}$ in the reference configuration)

$$\begin{aligned} \boldsymbol{F} &= \mathbf{P} + \boldsymbol{H}, & \mathbf{P} &= \nabla \boldsymbol{x} = \mathbf{I} - \boldsymbol{n} \otimes \boldsymbol{n}, & \boldsymbol{H} &= \nabla \boldsymbol{u}, \\ \boldsymbol{G} &= \mathring{\mathbf{K}} + \boldsymbol{\Phi}, & \boldsymbol{\Phi} &= \nabla \boldsymbol{\varphi}. \end{aligned}$$

We assume small displacements and small rotations:

$$|\boldsymbol{H}| << 1, \qquad |\boldsymbol{\Phi}| << 1. \tag{59}$$

$$\boldsymbol{S} = \mathring{\boldsymbol{S}} + \widetilde{\boldsymbol{S}}, \tag{60}$$

Special case: stress-free reference configuration

In this special case we have $\mathring{\boldsymbol{S}} = \mathbf{0}$ and $\mathring{\boldsymbol{M}} = 0$. In this case, the linearization of the equilibrium (49) and (50) yields

$$\begin{aligned} &\mathbf{P}(\operatorname{div} \boldsymbol{N} + \boldsymbol{b}) + \mathring{\mathbf{K}}(\operatorname{div} \boldsymbol{M} + \boldsymbol{c}) = \mathbf{0}, \\ &\mathbf{P} \operatorname{div}(\widetilde{\boldsymbol{M}} + \tilde{\boldsymbol{c}}) - \widetilde{\boldsymbol{N}} \cdot \mathring{\mathbf{K}} - \boldsymbol{n} \cdot \tilde{\boldsymbol{b}} = 0, \end{aligned}$$

Moreover, the incremental constitutive equations take the form

$$\begin{aligned} \boldsymbol{N} &= h\mathbb{C}\left[\frac{1}{2}(\mathring{\mathbf{P}}\boldsymbol{H} + \boldsymbol{H}^T\mathring{\mathbf{P}})\right] + \frac{h^3}{12}\mathring{\mathbf{K}}\mathbb{C}[\boldsymbol{H}^T\mathring{\mathbf{K}} + \boldsymbol{\Phi}] \\ \boldsymbol{M} &= \frac{h^3}{12}\mathbb{C}[\boldsymbol{H}^T\mathring{\mathbf{K}} + \boldsymbol{\Phi}] \end{aligned}$$

General case: shells with initial stress

$$\begin{aligned} \boldsymbol{N} &= \mathring{\mathbf{N}} + \boldsymbol{H}\mathring{\mathbf{N}} + h\mathbb{C}\left[\frac{1}{2}(\mathring{\mathbf{P}}\boldsymbol{H} + \boldsymbol{H}^T\mathring{\mathbf{P}})\right] + (\boldsymbol{\Phi} - \boldsymbol{H}\mathring{\mathbf{K}})\mathring{\mathbf{M}} + \frac{h^3}{12}\mathring{\mathbf{K}}\mathbb{C}[\boldsymbol{H}^T\mathring{\mathbf{K}} + \boldsymbol{\Phi}] \\ \boldsymbol{M} &= \mathring{\mathbf{M}} + \boldsymbol{H}\mathring{\mathbf{M}} + \frac{h^3}{12}\mathbb{C}[\boldsymbol{H}^T\mathring{\mathbf{K}} + \boldsymbol{\Phi}] \end{aligned}$$

Note: when performing the linearization, it is convenient to use the balance of forces in its form $\operatorname{div} \boldsymbol{S} + \boldsymbol{b} = \mathbf{0}$.

Remark: when writing the incremental equations, it is more convenient to linearize first $\operatorname{div} \boldsymbol{S} + \boldsymbol{b} = \mathbf{0}$, and only then perform the relevant projections of the resulting equations.

Appendix

Gradient and divergence on a surface

Give an surface $\mathcal{S}$ and a vector field $\boldsymbol{v}$, the gradient of $\boldsymbol{v}$ is a tensor field whose value at x is a map from T_x to V, defined as follows: the image of a vector $\boldsymbol{a}$ is computed by taking any smooth curve $a : \mathbb{R} \to \mathcal{S}$ such that $a(0) = x$ and $a'(0) = \boldsymbol{a}$, and by letting

$$\nabla \boldsymbol{v}[\boldsymbol{a}] = (\boldsymbol{v} \circ a)'(0). \tag{61}$$

This definition is independent on the particular curve chosen, as can be seen by working on a chart. The *covariant derivative* of $\boldsymbol{v}$ is

$$\nabla_s \boldsymbol{v} = \boldsymbol{P}\nabla\boldsymbol{v}. \tag{62}$$

Note that the covariant derivative is instead a linear map from T_x to T_x .

The divergence of a tangential vector field $\boldsymbol{a}$ is defined as $\operatorname{div}\boldsymbol{a} = \boldsymbol{P}\cdot\nabla\boldsymbol{a} = \boldsymbol{P}\nabla_s\boldsymbol{a}$, where $\boldsymbol{P}(x)$ is the projection of V to T_x.

The best way to introduce the divergence of a tensorial quantity $\boldsymbol{A}$ mapping T_x into some linear space is to consider first tensors of the form

$$\boldsymbol{A}(x) = \boldsymbol{v}\otimes\boldsymbol{a}(x), \tag{63}$$

where $\boldsymbol{v}$ is a constant vector. In this case, we have

$$\operatorname{div}\boldsymbol{A} = (\operatorname{div}\boldsymbol{a})\boldsymbol{v}. \tag{64}$$

It then follows that if $\boldsymbol{b}$ is another tangential tensor field, then

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{a} = \operatorname{div}\boldsymbol{a}\boldsymbol{a}\cdot\operatorname{div}\boldsymbol{A} + \nabla\boldsymbol{a}\cdot\boldsymbol{A}. \tag{65}$$

The effective Lamé constants for a linearly elastic shell

For a three-dimensional isotropic, linearly elastic material, the stress-strain relation has the form

$$\mathbf{S} = 2\mu\mathbf{E} + \lambda(\operatorname{tr}\mathbf{E})\mathbf{I}. \tag{66}$$

where μ and λ are the Lamé constants. As discussed in Ciarlet (2005)](https://doi.org/10.1007/s10659-005-4738-8), Sec. 4.5, a rigorous argument based on asymptotic analysis shows that the mechanical response of a shell-like homogeneous linearly elastic body of thickness h, made of a material obeying the stress-strain relation (66) is captured by a shell with Lamé constants

$$\widetilde{\mu} = \mu, \qquad \tilde{\lambda} = \frac{2\mu}{2\mu+\lambda}\lambda. \tag{67}$$

The divergence of a tensor field can be written as

$$\operatorname{div}\boldsymbol{T} = \boldsymbol{T}_{,\alpha}\boldsymbol{a}^{\alpha}. \tag{68}$$

In particular, $\boldsymbol{v}$ is a vector field and $\boldsymbol{T}$ is a tensor field, we have

$$\operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{T}) = \boldsymbol{v}\otimes\operatorname{div}\boldsymbol{T} + \boldsymbol{v}_{,\alpha}\otimes(\boldsymbol{T}\boldsymbol{a}^{\alpha}) = \boldsymbol{v}\otimes\operatorname{div}\boldsymbol{T} + (\boldsymbol{v}_{,\alpha}\otimes\boldsymbol{a}^{\alpha})\boldsymbol{T}^{T} = \boldsymbol{v}\otimes\operatorname{div}\boldsymbol{T} + \nabla\boldsymbol{v}\,\boldsymbol{T}^{T} \tag{69}$$

We obtain

$$\operatorname{div}(\boldsymbol{y}\otimes\boldsymbol{S}) = \boldsymbol{y}\otimes\operatorname{div}\boldsymbol{S} + \nabla\boldsymbol{y}\,\boldsymbol{S}^{T}, \qquad \operatorname{div}(\boldsymbol{d}\otimes\boldsymbol{M}) = \boldsymbol{d}\otimes\operatorname{div}\boldsymbol{M} + \nabla\boldsymbol{d}\,\boldsymbol{M}^{T}. \tag{70}$$

Hence,

$$\int_{\partial\mathcal{P}}(\boldsymbol{y}\otimes\boldsymbol{S}\boldsymbol{\nu} + \boldsymbol{n}\otimes\boldsymbol{M}\boldsymbol{\nu}) = \int_{\mathcal{P}}(\boldsymbol{y}\otimes\operatorname{div}\boldsymbol{S} + \nabla\boldsymbol{y}\,\boldsymbol{S}^{T} + \boldsymbol{n}\otimes\operatorname{div}\boldsymbol{M} + \nabla\boldsymbol{d}\,\boldsymbol{M}^{T}). \tag{71}$$

On introducing the decomposition

$$\boldsymbol{S} = \boldsymbol{d}\otimes\boldsymbol{S}^{T}\boldsymbol{d} + (\boldsymbol{I} - \boldsymbol{d}\otimes\boldsymbol{d})\boldsymbol{S} = \boldsymbol{d}\otimes\boldsymbol{q} + \boldsymbol{N}, \tag{72}$$

we can write

$$\mathrm{skw}\int_{\partial\mathcal{P}}(\boldsymbol{y}\otimes\boldsymbol{S}\boldsymbol{\nu}+\boldsymbol{d}\otimes\boldsymbol{M}\boldsymbol{\nu})=\mathrm{skw}\int_{\mathcal{P}}\Big[\boldsymbol{y}\otimes\operatorname{div}\boldsymbol{S}+\boldsymbol{d}\otimes(\operatorname{div}\boldsymbol{M}-\nabla\boldsymbol{y}^T\boldsymbol{q})-(\boldsymbol{N}\nabla\boldsymbol{y}^T+\boldsymbol{\nabla}\boldsymbol{d}^T)\Big].\quad(73)$$

An incomplete list of references:

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[3] P.M. Naghdi: The Theory of Shells and Plates, Handbuch der Physik VIa/2 p.425-640, Springer-Verlag, 1972.

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[5] A. Di Carlo, P. Podio-Guidugli, and W.O. Williams, Shells with thickness distension. International Journal of Solids and Structures 38(6-7):1201-1225 · February 2001

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1. Details to obtain the pointwise form of the balance equations↩

2. To prove this fact, one can use a coordinate system. Indeed $\boldsymbol{F}=\boldsymbol{y}_{,\alpha}\otimes\boldsymbol{a}^\alpha$. This yields $\nabla\boldsymbol{F}=\boldsymbol{y}_{,\alpha\beta}\otimes\boldsymbol{a}^\alpha\otimes\boldsymbol{a}^\beta+\boldsymbol{y}_{,\alpha}\otimes\boldsymbol{a}^\alpha_{,\beta}\otimes\boldsymbol{a}^\beta$. Since $\boldsymbol{y}_{,\alpha}\cdot\boldsymbol{n}=0$, we have $\nabla\boldsymbol{F}[\boldsymbol{n},\cdot,\cdot]=\boldsymbol{n}\cdot\boldsymbol{y}_{,\alpha\beta}\boldsymbol{a}^\alpha\otimes\boldsymbol{a}^\beta$, which is a tangential tensor.↩