Vitali Lemma (finite version). Given finite collection $\mathcal{C}$ of open balls $\{B_n\}$, there exists a finite subcollection $\{\overline{B}_n\}$ of *disjoint balls* such that $\cup_n B_n \subset \cup_n 3\overline{B}_n$.[1]

Proof. Define $\overline{B}_1$ as the largest ball in $\mathcal{C}$. Then, inductively, given $\overline{B}_1, \dots \overline{B}_k$ take $\overline{B}_{k+1}$ to be the largest ball in $\mathcal{C}$ that does not intersect $\cup_{j=1}^k \overline{B}_j$, and repeat the procedure until no such ball exists.

We now prove that each ball B_i of the original collection is contained in $\cup 3\overline{B}_n$. Indeed, B_i has a non-empty intersection with at least one ball $\overline{B}_k$ and this ball is larger than B_i (if not, B_i would have been selected in place of $\overline{B}_k$ at step $k-1$ in the above construction). Thus, $B_i \subset 3\overline{B}_k$

Vitali Lemma (infinite version). Given countable collection $\mathcal{C}$ of open balls $\{B_n\}$, such that $\sup_n \operatorname{diam} B_n < \infty$, there exists a finite subcollection $\{\overline{B}_n\}$ of *disjoint balls* such that $\cup_n B_n \subset \cup_n 3\overline{B}_n$.[2]

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[1] Given a ball B, we denote by $3B$ the ball whose radius is three times the radius of B.

[2] Given a ball B, we denote by $3B$ the ball whose radius is three times the radius of B.