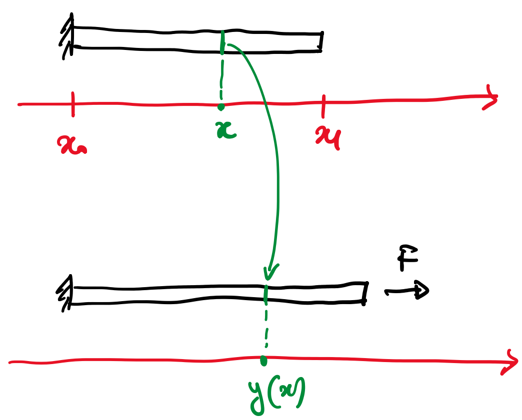
We are going to illustrate how stress and other relevant unknowns change when the material manifold is perturbed.

**Equilibrium problem.** For the sake of simplicity we consider the equilibrium problem for an elastic bar[1] clamped on one end, and subject to a traction on the opposite end.




The equilibrium state of the body is governed by the folliowing boundary-value problem:

$$\begin{aligned}
&\left.\begin{aligned}
&\sigma' + b = 0\\
&\sigma = \psi'(\lambda),\\
&\lambda = y',
\end{aligned}\right\}\ \text{in } (x_0, x_1)\\
&y(x_0) = Y_0,\\
&\sigma(x_1) = F.
\end{aligned}\tag{1}$$

**Perturbed problem.** We suppose that the domain $(x_0, x_1)$ undergoes an transformation which changes its extreme points into $(x_{0,\varepsilon}, x_{1,\varepsilon})$. We ask how such perturbation affects the solution.[2] Accordingly, we write the following

[1] By a bar we mean a body that is slender enough to be described as one dimensional, but such that the radius of gyration of the cross section of the bar is large enough compared with its length to prevent bending instabilities.

[2] To gain generality, we assume that the perturbation of the domain is accompanied by a perturbation of loads and constraints, but not of the free energy.

perturbed system:

$$\left.\begin{aligned} &\sigma_\varepsilon' + b_\varepsilon = 0 \\ &\sigma_\varepsilon = \widehat{\sigma}(\lambda_\varepsilon), \\ &\lambda_\varepsilon = y_\varepsilon', \end{aligned}\right\} \text{ in } (x_{0,\varepsilon}, x_{1,\varepsilon})$$
$$\begin{aligned} &y(x_{0,\varepsilon}) = Y_{0,\varepsilon}, \\ &\sigma(x_{1,\varepsilon}) = F_{0,\varepsilon}. \end{aligned} \tag{2}$$

In the above perturbation $\varepsilon$ is a parameter. We suppose that the perturbation is null for $\varepsilon = 0$, so that $x_0 = x_{0,0}$ and $x_1 = x_{1,0}$.

We define the infinitesimal increment of the traction as

$$\delta\sigma(x) = \left.\frac{\partial}{\partial\varepsilon}\right|_{\varepsilon=0} \sigma_\varepsilon(x), \tag{3}$$

as well as the infinitesimal increment of all other fields. We expect that the increments solve the following boundary value problem:

$$\left.\begin{aligned} &\delta\sigma' + \delta b = 0 \\ &\delta\sigma = \widehat{\psi}''(\lambda)\delta\lambda, \\ &\delta\lambda = \delta y', \end{aligned}\right\} \text{ in } (x_0, x_1)$$
$$\begin{aligned} &y(x_0) + y'(x_0)\delta x_0 = \delta Y_0, \\ &\sigma(x_1) + \sigma'(x_1)\delta x_1 = \delta F. \end{aligned} \tag{4}$$

This is indeed the case, as we show below.

**Change of domain.** To cope with the domain depending on $\varepsilon$ we reformulate the perturbed problem in the *initial domain* $(x_0, x_1)$. We do this by introducing a $\varepsilon$-parametrized class of diffeomorphisms $\varphi_\varepsilon : (x_0, x_1) \to (x_{0,\varepsilon}, x_{1,\varepsilon})$ such that

$$\varphi_\varepsilon(x_0) = x_{0,\varepsilon} \qquad \text{and} \qquad \varphi_\varepsilon(x_1) = x_{1,\varepsilon}. \tag{5}$$

We define the pullback of the traction as

$$\widetilde{\sigma}_\varepsilon(\widetilde{x}) = \sigma_\varepsilon(\varphi_\varepsilon(\widetilde{x})). \tag{6}$$

The equilibrium problem in the initial domain is

$$\left.\begin{aligned} &\widetilde{\sigma}_\varepsilon' + \varphi_\varepsilon'\widetilde{b}_\varepsilon = 0 \\ &\widetilde{\sigma}_\varepsilon = \psi'(\widetilde{\lambda}_\varepsilon), \\ &\widetilde{\lambda}_\varepsilon\varphi_\varepsilon' = \widetilde{y}_\varepsilon' - \varphi_\varepsilon', \end{aligned}\right\} \text{ in } (x_0, x_1)$$
$$\begin{aligned} &\widetilde{y}(x_0) = Y_{0,\varepsilon}, \\ &\widetilde{\sigma}(x_1) = F_{0,\varepsilon}. \end{aligned} \tag{7}$$

An important remark, at this point, is the following: ince the only requirement we impose on $\varphi_\varepsilon$ is that it maps the boundary of the unperturbed domain onto the boundary of the perturbed domain through (5), the way $\varphi_\varepsilon$ maps the interior of the unperturbed domain into the interior of the perturbed domain is not specified.

**Perturbation analysis.** We introduce the perturbation of the stress $\delta\widetilde{\sigma}(\widetilde{x}) = \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0}\widetilde{\sigma}_\varepsilon(\widetilde{x})$. We define the other perturbations in a similar manner. By linearizing (7) we obtain

$$\left.\begin{aligned} &\delta\widetilde{\sigma}' + \delta\widetilde{b} + b\delta\varphi' = 0 \\ &\delta\widetilde{\sigma} = EA\delta\widetilde{\lambda}, \\ &\delta\widetilde{\lambda} + \lambda\delta\varphi' = \delta\widetilde{y}' - \delta\varphi', \end{aligned}\right\} \text{ in } (x_0, x_1)$$
$$\begin{aligned} &\delta\widetilde{y}(x_0) = \delta Y_0, \\ &\delta\widetilde{\sigma}(x_1) = \delta F. \end{aligned} \tag{8}$$

We next define $\delta\sigma(x) = \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0}\sigma_\varepsilon(x)$. From this definition and from (6), it follows that

$$\delta\widetilde{\sigma} = \delta\sigma + \sigma'\delta\varphi. \tag{9}$$

Similar relations hold for all other increments, such as $\delta b$, etc.

Consider the first of (12). If we write this equation in terms of the increments $\delta\sigma$, $\delta b$ instead of $\delta\widetilde{\sigma}$, $\delta\widetilde{b}$, making use of (9) and of the other relations we obtain

$$(\delta\sigma + \sigma'\delta\varphi)' + \delta b + b'\delta\varphi + b\delta\varphi' = 0. \tag{10}$$

Using the equation $\sigma' + b = 0$ one can check that most of the terms in (10) cancel out, and

$$\delta\sigma' + \delta b = 0. \tag{11}$$

$$\left.\begin{aligned} &\delta\sigma' + \delta b = 0 \\ &\delta\sigma = \psi''(\sigma)\delta\lambda, \\ &\delta\lambda = \delta y', \end{aligned}\right\} \text{ in } (x_0, x_1)$$
$$\begin{aligned} &\delta y(x_0) + y'(x_0)\delta x_0 = \delta Y_0, \\ &\delta\sigma(x_1) + \sigma'(x_1)\delta x_1 = \delta F. \end{aligned} \tag{12}$$

**Important remark.** It is worth noticing that the result does not depend on the maps used to generate the perturbed domain.

For a similar result in the three-dimensional setting see here. In this document, the perturbed boundary value problem is obtained using the principle of virtual work.