Consider the boundary-value problem of nonlinear elastostatics:

$$\left.\begin{aligned} &\operatorname{div}\mathbf{S}+\mathbf{b}=0\\ &\mathbf{S}=\psi'(\mathbf{F}),\\ &\mathbf{F}=\nabla\mathbf{y}, \end{aligned}\right\}\ \text{in } \mathcal{B} \qquad \begin{aligned} &\mathbf{y}=\mathbf{y}_0, \quad \text{on } \partial_c\mathcal{B}\\ &\mathbf{S}\mathbf{n}=\mathbf{s}_0 \quad \text{on } \partial_f\mathcal{B}. \end{aligned} \tag{P}$$

We suppose that the domain $\mathcal{B}$ undergoes a perturbation $\mathcal{B}\to\mathcal{B}_\varepsilon$, where ε is a small parameter, and we consider the perturbed problem

$$\left.\begin{aligned} &\operatorname{div}\mathbf{S}_\varepsilon+\mathbf{b}_\varepsilon=0\\ &\mathbf{S}_\varepsilon=\psi'(\mathbf{F}_\varepsilon),\\ &\mathbf{F}_\varepsilon=\nabla\mathbf{y}_\varepsilon, \end{aligned}\right\}\ \text{in } \mathcal{B}_\varepsilon \qquad \begin{aligned} &\mathbf{y}_\varepsilon=\mathbf{y}_{0,\varepsilon}, \quad \text{on } \partial_c\mathcal{B}_\varepsilon\\ &\mathbf{S}_\varepsilon\mathbf{n}=\mathbf{s}_\varepsilon \quad \text{on } \partial_f\mathcal{B}_\varepsilon. \end{aligned} \tag{P$_\varepsilon$}$$

Given any ε-dependent field ϕ_ε on $\mathcal{B}_\varepsilon$, and a point $x\in\mathcal{B}$, we define the increment of $\phi_\varepsilon(x)$ as[1]

$$\delta\phi(x)=\left.\frac{\partial}{\partial\varepsilon}\right|_{\varepsilon=0}\phi_\varepsilon(x). \tag{1}$$

We are going to show that the increments are solution of the following BPV:

$$\left.\begin{aligned} &\operatorname{div}\delta\mathbf{S}+\delta\mathbf{b}=0\\ &\delta\mathbf{S}=\psi''(\mathbf{F})[\delta\mathbf{F}],\\ &\delta\mathbf{F}=\nabla\delta\mathbf{y}, \end{aligned}\right\}\ \text{in } \mathcal{B} \qquad \begin{aligned} &\delta\mathbf{y}+\mathbf{F}\delta\boldsymbol{\varphi}=\delta\mathbf{y}_0, \quad \text{on } \partial_c\mathcal{B}\\ &\delta\mathbf{S}\mathbf{n}+\nabla\mathbf{S}[\delta\boldsymbol{\varphi}]\mathbf{n}+\mathbf{S}\delta\mathbf{n}=\delta\mathbf{s}_0 \quad \text{on } \partial_f\mathcal{B}. \end{aligned} \tag{P$_\varepsilon$}$$

For simplicity, we prove this result in the case when $\mathbf{s}_0$ vanishes.

As a start, we choose an *arbitrary* test function $\mathbf{v}$ defined on $\mathcal{B}$ such that $\mathbf{v}=0$ on $\partial_c\mathcal{B}$. Then, we let $\mathbf{v}_\varepsilon$ be the unique test function on $\mathcal{B}_\varepsilon$ such that $\mathbf{v}_\varepsilon(\varphi_\varepsilon(x))=\mathbf{v}(x)$. Then,

$$\int_{\mathcal{B}_\varepsilon}\mathbf{S}_\varepsilon\cdot\nabla\mathbf{v}_\varepsilon=\int_{\mathcal{B}_\varepsilon}\mathbf{b}_\varepsilon\cdot\mathbf{v}_\varepsilon. \tag{2}$$

As done in the one-dimensional case, we change the domain of integration by considering a diffeomorphism $\boldsymbol{\varphi}_\varepsilon:\mathcal{B}\to\mathcal{B}_\varepsilon$, and by defining the functions

$$\widetilde{\mathbf{S}}_\varepsilon(x)=\mathbf{S}_\varepsilon(\boldsymbol{\varphi}_\varepsilon(x)), \qquad \widetilde{\mathbf{b}}_\varepsilon(x)=\mathbf{b}_\varepsilon(\boldsymbol{\varphi}_\varepsilon(x)), \tag{3}$$

we obtain

$$\int_{\mathcal{B}}\widetilde{\mathbf{S}}_\varepsilon(\operatorname{Cof}\nabla\boldsymbol{\varphi}_\varepsilon)\cdot\nabla\mathbf{v}=\int_{\mathcal{B}}(\det\nabla\boldsymbol{\varphi}_\varepsilon)\widetilde{\mathbf{b}}_\varepsilon\cdot\mathbf{v} \qquad \forall\mathbf{v}=0 \text{ on } \partial_c\mathcal{B}. \tag{4}$$

[1]This quantity is well defined since $\mathcal{B}$ is an open set.

It is important to keep in mindi that $\boldsymbol{\varphi}_\varepsilon$ is not defined uniquely. What matters is that it maps $\mathcal{B}$ into $\mathcal{B}_\varepsilon$.

We differentiate with respect to ε at $\varepsilon = 0$, and use the relation

$$\delta\widetilde{\mathbf{S}} = \delta\mathbf{S} + \nabla\mathbf{S}[\delta\boldsymbol{\varphi}], \tag{5}$$

as well as the analogous relation of $\delta\mathbf{b}$ to obtain

$$\int_{\mathcal{B}} \delta\mathbf{S}\cdot\nabla\boldsymbol{v}+\nabla\mathbf{S}[\delta\boldsymbol{\varphi}]\cdot\nabla\mathbf{v}+\mathbf{S}\delta(\mathrm{Cof}\,\nabla\boldsymbol{\varphi})\cdot\nabla\mathbf{v} = \int_{\mathcal{B}} \delta\mathbf{b}\cdot\mathbf{v}+(\nabla\mathbf{b}\delta\boldsymbol{\varphi})\cdot\mathbf{v}+\mathbf{b}\cdot(\delta(\det\nabla\boldsymbol{\varphi})\mathbf{v}). \tag{6}$$

Now we can essentially repeat what has been done in the the one-dimensional case. Using integration by parts and the identity $\mathrm{div}\,\delta(\mathrm{Cof}\,\nabla\boldsymbol{\varphi}) = \mathbf{0}$, we obtain

$$\begin{aligned} &\int_{\mathcal{B}} [\delta\mathbf{S} + \nabla\mathbf{S}[\delta\boldsymbol{\varphi}] + \mathbf{S}\delta(\mathrm{Cof}\,\nabla\boldsymbol{\varphi})] \cdot \nabla\mathbf{v} \\ &= -\int_{\mathcal{B}} (\mathrm{div}\,\delta\mathbf{S}+(\nabla\,\mathrm{div}\,\mathbf{S})\delta\boldsymbol{\varphi})\cdot\mathbf{v}+\mathrm{div}\,\mathbf{S}\cdot(\delta(\det\nabla\boldsymbol{\varphi})\mathbf{v})+\int_{\partial\mathcal{B}} [\delta\mathbf{S} + \nabla\mathbf{S}[\delta\boldsymbol{\varphi}] + \mathbf{S}\delta(\mathrm{Cof}\,\nabla\boldsymbol{\varphi})]\,\mathbf{n}\cdot\mathbf{v}. \end{aligned} \tag{7}$$

We substitute (7) into (6). Then we observe that thanks to the equilibrium equation $\mathrm{div}\,\mathbf{S} + \mathbf{b} = \mathbf{0}$, several terms cancel out, and the final result is

$$\int_{\mathcal{B}} (\mathrm{div}\,\delta\mathbf{S} + \delta\mathbf{b}) \cdot \mathbf{v} + \int_{\partial\mathcal{B}} [\delta\mathbf{S} + \nabla\mathbf{S}[\delta\boldsymbol{\varphi}] + \mathbf{S}\delta(\mathrm{Cof}\,\nabla\boldsymbol{\varphi})]\,\mathbf{n} \cdot \mathbf{v} = 0. \tag{8}$$

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