J. Mech. Phys. Solids 150 (2021) 104353

Contents lists available at ScienceDirect

Journal of the Mechanics and Physics of Solids

journal homepage: www.elsevier.com/locate/jmps

Applications of the J-integral to dynamical problems in geotechnical engineering

J. Garcia-Suarez *, D. Asimaki, M. Ortiz

Division of Engineering and Applied Science, California Institute of Technology, 1200 E. California Blvd., Pasadena, CA 91125, United States of America

ARTICLE INFO

Keywords:
Conservation laws
Path-independent integrals
J integral
Geotechnical engineering
Seismic pressures

ABSTRACT

We formulate a path-independent J-integral for the elastodynamic problem expressed in the frequency domain. We show that the path-independence of the integral can be exploited in order to derive *ansatz*-free identities and rigorous inequalities in certain problems arising in geotechnical engineering. By way of illustration, we specifically consider the problem of assessing seismic pressures on retaining walls. We show that the bounds for the earth thrust derived from the frequency-domain dynamic J-integral improve upon previous heuristic and conjectured bounds.

1. Introduction

The energy–momentum tensor is a fundamental concept in the field theories of physics (Marsden and Hughes, 1994). Path-independent integrals follow from the divergence-free property of the energy–momentum tensor. Their application to solid mechanics problems dates back to Eshelby (1956). Subsequently, Eshelby's work was further specialized to interfaces and surfaces of discontinuity (Abeyaratne and Knowles, 2006), fracture mechanics (Rice, 1968) dislocations (Rice, 1985) and other areas of application. In the context of elastostatics, Knowles and Sternberg (1972) elucidated the connection between path-independent integrals and Noether's Theorem (Byers, 1998). Their theoretical framework was subsequently extended to elastodynamics by Fletcher (1976).

In this paper, we formulate path-independent integrals for the elastodynamic problem expressed in the frequency domain. We show how such path-independent integrals can be applied to derive rigorous identities and bounds in certain problems arising in geotechnical engineering. We illustrate the approach by means of the problem of assessing earthquake-induced pressures on retaining walls (Wood, 1973). We specifically consider an unyielding retaining wall supported on rigid bedrock and a soil backfill held by the wall (Fig. 1). The system is assumed to undergo plane-strain deformation and it deforms under the action of a prescribed SV-wave. The soil is modeled as homogeneous, isotropic and linear–elastic. An slightly different configuration wherein the soil can slide over the wall with no friction ("smooth wall") is also considered in the literature, and an exact solution in the frequency and horizontal wavenumber domain has been provided recently (Garcia-Suarez and Asimaki, 2020).

This problem has been studied extensively by recourse to sundry simplifying assumptions. To name some of the most relevant contributions: Matsuo and Ohara (1960) derived the first closed-form simplified solution based on an *ansatz* of no horizontal displacement anywhere in the soil domain; Arias et al. (1981) simplified the problem by considering spring-like behavior in the soil instead of full continuum elasticity; Veletsos and Younan (1994a) simplified the problem based on an *ansatz* of zero vertical normal stress in the soil; Kloukinas et al. (2012) combined Younan and Veletsos' previous work with Vlasov's method (Vlasov,

* Corresponding author.
E-mail address: ajgarcia@caltech.edu (J. Garcia-Suarez).

https://doi.org/10.1016/j.jmps.2021.104353
Received 10 December 2020; Received in revised form 1 February 2021; Accepted 2 February 2021
Available online 14 February 2021

1966) to approximate the partial differential equations of the problem as a single ordinary differential equation representing the fundamental mode of the system; their work has recently been extended to account for inhomogeneous backfills (Brandenberg et al., 2017). A number of simplified models (Ostadan, 2005) supported by numerical calculations have been proposed by the National Earthquake Hazard Reduction Program recommendations (NEHRP, 2001).

In contrast to these heuristics, the identities and bounds derived in the present work are *ansatz*-free, i. e., follow directly from the full formulation of the problem without recourse to simplifying assumptions. We also note that the application of frequency-domain dynamic J-integral extends to a range of other problems, including layered media. As in static elasticity, it is also possible to derive other path independent integrals from the frequency-domain dynamic Eshelby tensor, in direct analogy to the classical L and M integrals. These additional integrals further extend the scope of the method, e.g., to problems of overturning foundations and scattering of seismic waves by cavities.

There are still a number of intrinsic limitations in the way the soil is modeled: On one hand, granular media as soil is known to fluidizate for some combinations of load intensity and vibration frequency (Richards et al., 1990). Likewise, some authors point out that the soil should be considered (apart from heterogeneous) not isotropic but anisotropic due to the effect of self-weight pressure on its stiffness (Gibson, 1974). Finally, recall that the response of soft soils during high-magnitude earthquakes is neither linear nor elastic (Kramer, 1996). Notwithstanding these caveats, we consider that the content of the article already covers many cases of importance in engineering practice.

The paper is organized as follows Section 2 provides a derivation of the frequency-domain path-independent J-integral that provides the basis for subsequent applications. In Section 3, this J integral is related to the earth thrust acting on the retaining wall in a certain configuration and an upper bound is derived in terms of the far-field. Section 4 is concerned with the verification of the bounds using numerical simulations. Finally, Section 5 states the main conclusions and discusses avenues for future research.

2. Elastodynamics in the frequency domain

We consider a domain B of spatial dimension n referred to a Cartesian reference frame $x = [x_1, \ldots, x_n]^\top$. We seek wave solutions $u_i(x,t)$ over the entire time domain $-\infty \leq t \leq +\infty$ of the Cauchy–Navier elastodynamic problem (Einstein indices notation is in effect)

$$\mu u_{i,jj} + (\lambda+\mu) u_{j,ji} + f_i = \rho \ddot{u}_i, \quad \text{in } B, \tag{1a}$$

$$u_i = g_i \quad \text{on } \partial B_D\,, \tag{1b}$$

$$\lambda u_{k,k} n_i + \mu(u_{i,j} + u_{j,i}) n_j = h_i \quad \text{on } \partial B_N\,, \tag{1c}$$

where ∂B_D (∂B_N) is the contour of the domain where displacements (tractions) boundary conditions are applied, g_i (h_i) being the imposed value, f_i represents a body force acting along the ith direction, μ and λ are the Lamé material constants and ρ represents the density (homogeneous isotropic material).

An application of the Fourier transform gives

$$\mu \hat{u}_{i,jj} + (\lambda+\mu)\hat{u}_{j,ji} + \hat{f}_i + \rho\omega^2 \hat{u}_i = 0, \quad \text{in } B, \tag{2a}$$

$$\hat{u}_i = \hat{g}_i \quad \text{on } \partial B_D\,, \tag{2b}$$

$$\lambda \hat{u}_{k,k} n_i + \mu(\hat{u}_{i,j} + \hat{u}_{j,i}) n_j = \hat{h}_i \quad \text{on } \partial B_N\,, \tag{2c}$$

where a superposed $\hat{}$ denotes the Fourier transform of the function with respect to time.

2.1. A frequency-domain dynamic path-independent integral

In the absence of body forces, the energy–momentum tensor corresponding to field equations (2) follows from standard theory (Maugin, 2016; Gurtin, 1999) as

$$M_{ij} = \left(W(\hat{\epsilon}) - \rho\omega^2 \hat{u}_i \hat{u}_i^* \right)\delta_{ij} - \hat{u}_{k,i}\hat{\sigma}^*_{k,j}. \tag{3}$$

where

$$W(\hat{\epsilon}) = \frac{\lambda}{2}|\hat{\epsilon}_{kk}|^2 + \mu \hat{\epsilon}_{ij}\hat{\epsilon}^*_{ij}, \tag{4}$$

is the strain–energy density,

$$\hat{\epsilon}_{ij} = \frac{1}{2}(\hat{u}_{i,j} + \hat{u}_{j,i}), \tag{5}$$

is the Fourier transform of the strain tensor, and

$$\hat{\sigma}_{ij} = \lambda \hat{\epsilon}_{kk}\delta_{ij} + 2\mu\hat{\epsilon}_{ij}, \tag{6}$$

is the Fourier transform of the stress tensor.

We verify directly that

$$M_{ij,j} = 0 \tag{7}$$

that is, the energy–momentum tensor is divergence-free.

Integrating over a subdomain E of B with boundary Γ,

$$\mathrm{J}_i = \int_E M_{ij,j}\, dV = \int_\Gamma M_{ij} n_j\, dS = 0. \tag{8}$$

If E is contained between to surfaces Γ_1 and Γ_2, then

$$\int_{\Gamma_1} M_{ij} n_j\, dS = \int_{\Gamma_2} M_{ij} n_j\, dS, \tag{9}$$

with the appropriate choice of normals, which establishes the path-independence property of M. In two dimensions, the boundaries Γ_1 and Γ_2 are oriented closed curves and the integrals thus reduce to contour integrals.

2.1.1. Variational/configurational formulation of the elastodynamics Helmholtz problem

A relation between the energy–momentum tensor and configurational forces may be derived from the action integral as follows. For fixed ω, the solutions of Eq. (1) are stationary points of the action

$$A[u;\omega] = \int_B L(x,\hat{u},\nabla\hat{u})\, dV + \int_{B_N} \hat{h}_i \hat{u}_i^*\, dS \tag{10}$$

subject to the essential boundary conditions, Eq. (2b), where

$$L(x,\hat{u},\nabla\hat{u}) = \frac{\rho(x)\omega^2}{2}\hat{u}_i\hat{u}_i^* - W(x,\hat{\epsilon}) + \hat{f}_i(x)\hat{u}_i^* \tag{11}$$

is a Lagrangian density in the frequency domain and we allow the material to be inhomogeneous. The action (10) is defined over waves of infinite duration and can be derived directly from the standard time-domain elastodynamic action using Parseval's identity. The stationarity of (10) demands

$$\delta A[u\,,\eta;\omega] = \int_B \left(\frac{\partial L}{\partial \hat{u}_{i,j}}\eta_{i,j} + \frac{\partial L}{\partial \hat{u}_i}\eta_i \right) dV + \int_{\partial B_N} \hat{t}_i\, \hat{u}_i\, dS = 0\,, \tag{12}$$

for every admissible variation η satisfying homogeneous displacement boundary conditions. Consider now rearrangements of the heterogeneity of the form

$$L_\epsilon(x,\hat{u},\nabla\hat{u}) = L(x-\epsilon\eta,\hat{u},\nabla\hat{u}), \tag{13}$$

where η represents an admissible displacement. Let u_ϵ be the stationary point of the action A_ϵ. The rate of change of the action due to the rearrangement then follows as

$$\left.\frac{dA_\epsilon}{d\epsilon}\right|_{\epsilon=0} = \int_B \left(\frac{\partial L}{\partial \hat{u}_{i,j}}\eta_{i,j} + \frac{\partial L}{\partial \hat{u}_i}\eta_i - \left.\frac{\partial L}{\partial x_i}\right|_{\mathrm{exp}} \eta_i \right) dV + \int_{\partial B_N} \hat{t}_i\, \eta_i\, dS, \tag{14}$$

where "exp" signifies that the partial derivatives are taken with respect to the explicit dependence of L with respect to x. By the stationarity condition (12), Eq. (14) reduces to

$$\left.\frac{dA_\epsilon}{d\epsilon}\right|_{\epsilon=0} = -\int_B \left.\frac{\partial L}{\partial x_i}\right|_{\mathrm{exp}} \eta_i dV \equiv \int_B J_i \eta_i dV, \tag{15}$$

where J_i is the local configurational force conjugate to η_i. A straightforward calculation additionally gives the identity

$$J_i = -\left.\frac{\partial L}{\partial x_i}\right|_{\mathrm{exp}} = M_{ij,j}, \tag{16}$$

which establishes the sought relation between configuration forces and the energy–momentum tensor. The relation between J_i, Eq. (8), and J_i is the one of integral and integrand:

$$\mathrm{J}_i = \int_E J_i\, dV = \int_\Gamma M_{ij} n_j\, dS = 0. \tag{17}$$

3. Application to retaining walls

As an application of the theory, we consider a rigid retaining wall supported on rigid bedrock and a soil backfill held by the wall, Fig. 1. The system is assumed to undergo plane-strain deformation and responds under the action of a prescribed SV-wave. We assume the roughness of the wall prevents relative displacements of the soil and to induce shear stresses at the wall-backfill and bedrock-backfill interfaces. In terms of the relative displacements with respect to the base motion, the problem is equivalent to a fixed base and horizontal body forces

$$f = -\rho\varpi^2 X_g e^{\mathrm{i}\varpi t}, \tag{18}$$

where X_g is the amplitude of the ground motion and ϖ its frequency. We adapt the notation to the particular geometry under consideration and refer to the horizontal component of the relative displacement as $\hat{u}$, to the vertical component as $\hat{v}$ and, likewise, we set $x_1 \equiv x$ for the horizontal coordinate and $x_2 \equiv z$ for the vertical. We assume the soil to be homogeneous with density ρ, linear–elastic (Lamé constants μ and λ and Poisson's ratio $\nu = \lambda/2(\lambda+\mu)$). We additionally account for damping by recourse to the correspondence principle in frequency domain (Ben-Menahem and Singh, 2012) and a complex shear modulus $\mu(1+\mathrm{i}\delta_d)$, where i is the imaginary unit number and δ_d is the damping coefficient.

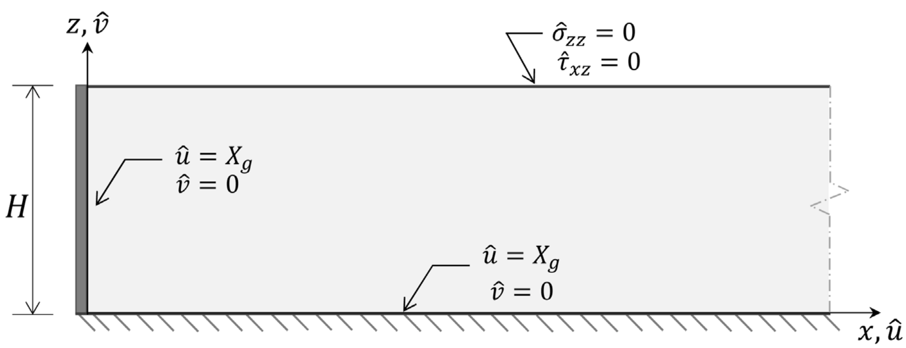

Fig. 1. Schematic of a rigid retaining wall supported on rigid bedrock and a soil backfill held by the wall. The system is assumed to undergo plane-strain deformation and responds under the action of a prescribed SV-wave.

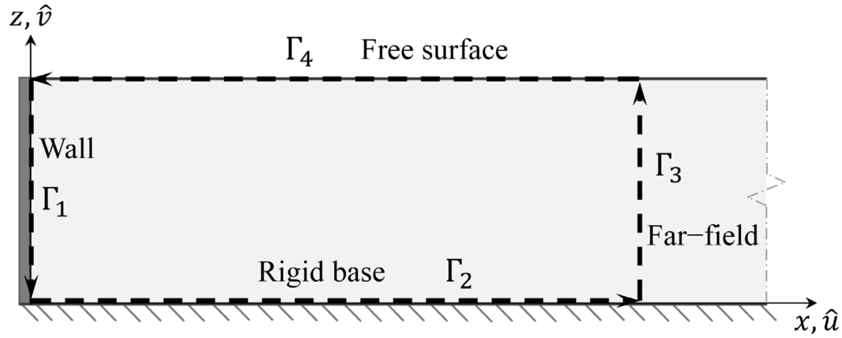

Fig. 2. Contour used to evaluate the J integral.

3.1. The identity supplied by the J-integral

The frequency-domain J-integral derived in the foregoing supplies useful identities whose evaluation does not require knowledge of the full exact solution. One such identity can be obtained by considering the contour depicted in Fig. 2. The first segment of the contour, Γ_1, runs over the wall, from top to bottom, Γ_2 stretches over the rigid bedrock Γ_3 samples the far-field response and Γ_4 extends over the free surface of the backfill.

Evaluating the integral Eq. (8) with $i = 1$ over the contour Γ gives

$$\begin{aligned} \mathrm{J}_1 = & \int_0^H \frac{1}{2}\left[(\lambda+2\mu)\left|\frac{\partial \hat{u}}{\partial x}\right|^2 + \mu\left|\frac{\partial \hat{v}}{\partial x}\right|^2 + \rho\omega^2 X_g^2\right]_{x=0} dz \\ & - \int_0^H \frac{1}{2}\left[\rho\omega^2|\hat{u}|^2 - \mu\left|\frac{\partial \hat{u}}{\partial z}\right|^2\right]_{x\to\infty} dz = 0\,, \end{aligned} \tag{19}$$

Conveniently, neither the free-surface nor the rigid base contribute to this expression. Simplifying, we obtain the identity

$$\int_0^H \frac{1}{2}\left[(\lambda+2\mu)\left|\frac{\partial \hat{u}}{\partial x}\right|^2 + \mu\left|\frac{\partial \hat{v}}{\partial x}\right|^2\right]_{x=0} dz = \mathrm{J}_\infty, \tag{20}$$

where

$$\mathrm{J}_\infty = \int_0^H \frac{1}{2}\left[\rho\omega^2(|\hat{u}|^2 - X_g^2) - \mu\left|\frac{\partial \hat{u}}{\partial z}\right|^2\right]_{x\to\infty} dz\,. \tag{21}$$

The right-hand side of (20) represents the action supplied by the far-field, whereas the left-hand side samples conditions at the wall-backfill interface. We recall that far-field displacement is Kramer (1996)

$$\hat{u} = X_g \frac{\cos(r\xi)}{\cos(r)}\,, \tag{22}$$

where

$$\xi = 1 - \frac{z}{H}, \qquad r = \frac{\varpi H}{c_s\sqrt{1+\mathrm{i}\delta_d}}, \tag{23}$$

whereupon (21) evaluates to

$$\mathrm{J}_\infty = \left| \frac{2\sinh(\delta_d r) - (\delta_d + 2\mathrm{i}) r \cosh(\delta_d r) - (\delta_d + 2\mathrm{i}) r \cos(2r) + 2\mathrm{i} \sin(2r)}{(\delta_d + 2\mathrm{i})(\cosh(\delta_d r) + \cos(2r)) r^3} \right|, \tag{24}$$

wherein the approximation $1/\sqrt{1+\mathrm{i}\delta_d} \approx 1 - \mathrm{i}\delta_d/2$ has also been accounted for, since it is customarily assumed that $\delta_d \ll 1$.

3.2. Bounds for the earth thrust

The earth thrust can be evaluated using the same frequency-domain framework developed earlier. The thrust has two components: horizontal, $\hat{\mathrm{Q}}_x$, and vertical, $\hat{\mathrm{Q}}_z$, given by

$$\hat{\mathrm{Q}}_x = \int_0^H \left[\hat{\sigma}_{xx}\right]_{x=0} dz, \qquad \hat{\mathrm{Q}}_z = \int_0^H \left[\hat{\tau}_{xz}\right]_{x=0} dz. \tag{25}$$

Since the soil is bonded to the rigid wall, we have

$$\left[\frac{\partial \hat{v}}{\partial z}\right]_{x=0} = 0, \qquad \left[\frac{\partial \hat{u}}{\partial z}\right]_{x=0} = 0. \tag{26}$$

By virtue of these constraints, the wall tractions simplify to

$$\left[\hat{\sigma}_{xx}\right]_{x=0} = (\lambda + 2\mu) \left[\frac{\partial \hat{u}}{\partial x}\right]_{x=0}, \qquad \left[\hat{\tau}_{xz}\right]_{x=0} = \mu \left[\frac{\partial \hat{v}}{\partial x}\right]_{x=0}, \tag{27}$$

and the thrust components (25) to

$$\hat{\mathrm{Q}}_x = \int_0^H (\lambda + 2\mu) \left[\frac{\partial \hat{u}}{\partial x}\right]_{x=0} dz, \qquad \hat{\mathrm{Q}}_z = \int_0^H \mu \left[\frac{\partial \hat{v}}{\partial x}\right]_{x=0} dz. \tag{28}$$

An application of the Cauchy–Schwarz inequality gives the estimates

$$|\hat{\mathrm{Q}}_x|^2 \leq H \int_0^H \left|(\lambda + 2\mu) \left[\frac{\partial \hat{u}}{\partial x}\right]_{x=0}\right|^2 dz, \qquad |\hat{\mathrm{Q}}_z|^2 \leq H \int_0^H \left|\mu \left[\frac{\partial \hat{v}}{\partial x}\right]_{x=0}\right|^2 dz. \tag{29}$$

But, the J-integral identity (20) requires

$$\int_0^H \frac{1}{2} \left[(\lambda + 2\mu) \left|\frac{\partial \hat{u}}{\partial x}\right|^2\right]_{x=0} \leq \mathrm{J}_\infty, \qquad \int_0^H \frac{1}{2} \left[\mu \left|\frac{\partial \hat{v}}{\partial x}\right|^2\right]_{x=0} dz \leq \mathrm{J}_\infty, \tag{30}$$

which, in view of (29), bounds the earth thrust components as

$$|\hat{\mathrm{Q}}_x|^2 \leq 2H(\lambda + 2\mu)\mathrm{J}_\infty, \qquad |\hat{\mathrm{Q}}_z|^2 \leq 2H\mu\mathrm{J}_\infty. \tag{31}$$

Combining the two previous results, it follows that the total magnitude of the thrust satisfies $|\hat{\mathrm{Q}}|^2 = |\hat{\mathrm{Q}}_x|^2 + |\hat{\mathrm{Q}}_z|^2 \leq 2H(\lambda + 3\mu)\mathrm{J}_\infty$. Expressing this result in dimensionless form:

$$\frac{|\hat{\mathrm{Q}}|}{\rho \ddot{X}_g H^2} \leq \sqrt{(1 + c^2)\mathrm{J}_\infty}, \tag{32}$$

where $\ddot{X}_g = \varpi^2 X_g$ and $c = c(\nu) = \sqrt{2(1-\nu)/(1-2\nu)}$ is the ratio between the P-wave velocity and the S-wave velocity in the material.

3.3. Derivation of approximate closed-form expression

Using Eq. (22) to evaluate the integrals one obtains Eq. (24). Alternatively, we may note that

$$\frac{2\sinh(\delta_d r) - (\delta_d + 2\mathrm{i}) r \cosh(\delta_d r) - (\delta_d + 2\mathrm{i}) r \cos(2r) + 2\mathrm{i} \sin(2r)}{(\delta_d + 2\mathrm{i})(\cosh(\delta_d r) + \cos(2r)) r^3} = \frac{r - \tan(r)}{r^3} + \mathcal{O}(\delta_d). \tag{33}$$

Since we assume that $\delta_d \ll 1$, we can approximate the expression by means of the first term. The order of approximation remains unchanged in the presence of damping. The result is

$$\frac{|\hat{\mathrm{Q}}|}{\rho \ddot{X}_g H^2} \leq \sqrt{1 + c^2} \sqrt{\left|\frac{r - \tan(r)}{r^3}\right|}, \tag{34}$$

which fits closely the results obtained from direct integration of J_∞, (cf. (21) and Mathematica (Wolfram, 2000) notebook in Supplementary Material). In the quasistatic limit of $r \to 0$, corresponding to long-wavelength, low-frequency loading, (34) further reduces to

$$\frac{|\hat{\mathrm{Q}}^{qs}|}{\rho \ddot{X}_g H^2} \leq \frac{\sqrt{1 + c^2}}{\sqrt{3}} = \sqrt{\frac{3 - 4\nu}{3(1 - 2\nu)}}. \tag{35}$$

4. Verifications

We assess the performance of the bounds derived in the foregoing by means of selected comparisons to finite-element solutions obtained using the commercial package *Abaqus CAE* (Simulia, 2010).

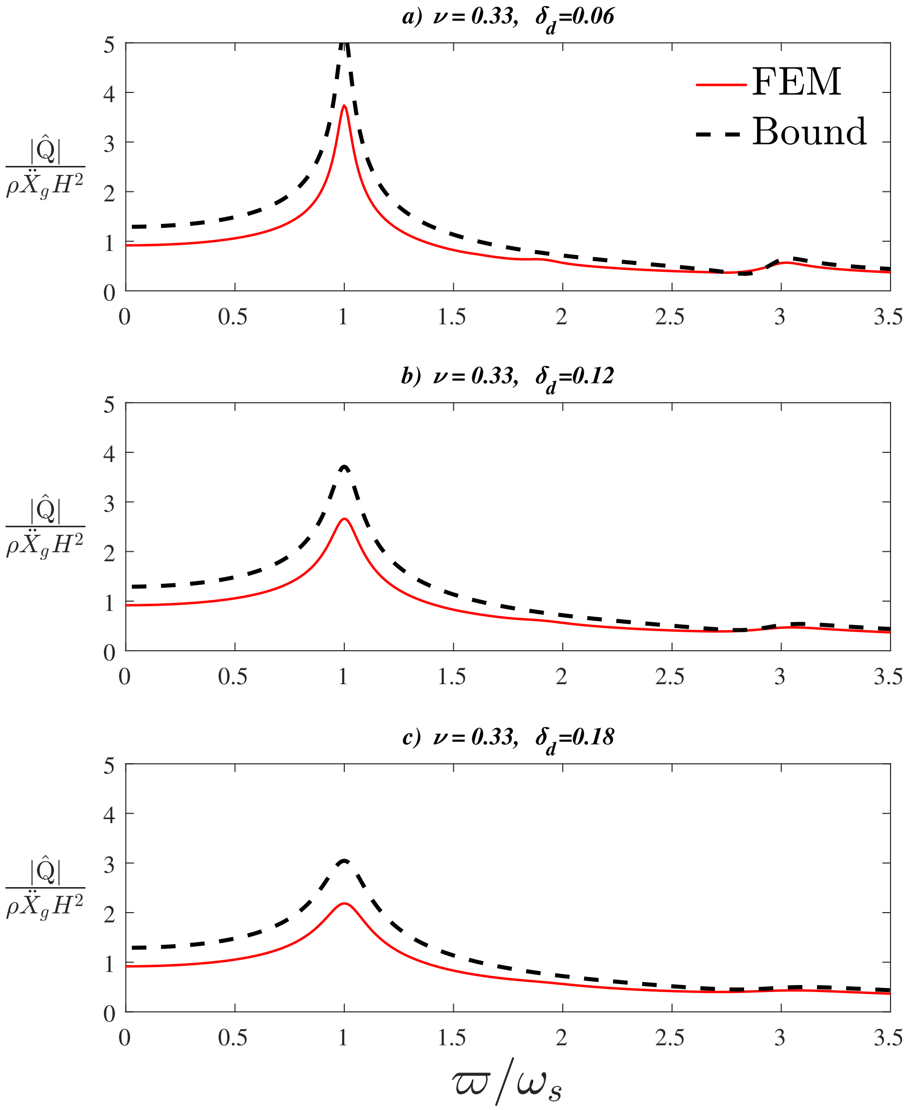

Fig. 3. Numerical results compared to bound Eq. (34). Damping sensitivity for homogeneous layer with $\nu = 0.33$.

4.1. Dynamic bound

We introduce structural damping in the *Abaqus* simulations, which, by the corresponding principle, amounts to effecting the substitution $\mu \rightarrow \mu(1 + i\delta_d)$ (cf., e.g., Ben-Menahem and Singh (2012), section 10.3.1.1). The abscissa of the dynamic plots correspond to the excitation frequency, henceforth denoted ϖ. We additionally introduce the characteristic frequency $\omega_s = \pi c_s/2H$, corresponding to the fundamental frequency of the far-field, for purposes of normalization.

Fig. 3 evinces the good accuracy of the dynamic bound over a range of damping, especially at high frequencies and high compressibility, Fig. 4. We note that the resonance peaks in the thrust lie at the same frequency for which the soil column in the far-field resonates, as required. This property, which follows directly from Eq. (34), underscores the influence of on-the-wall boundary conditions on the response of the soil next to the structure. For instance, if the soil slides over the wall there are other natural frequencies for the thrust that do not correspond to the far-field (Garcia-Suarez and Asimaki, 2020).

4.2. Quasi-static bound

Similar trends are observed in the quasistatic results, Fig. 5. This limiting case corresponds to wavelengths much larger than the height of the wall. Such conditions are often realized in practice for common soil properties, earthquake frequency content, and structural sizes. Remarkably, the bound diverges, and becomes trivial, in the incompressible limit. By contrast, in the intermediate

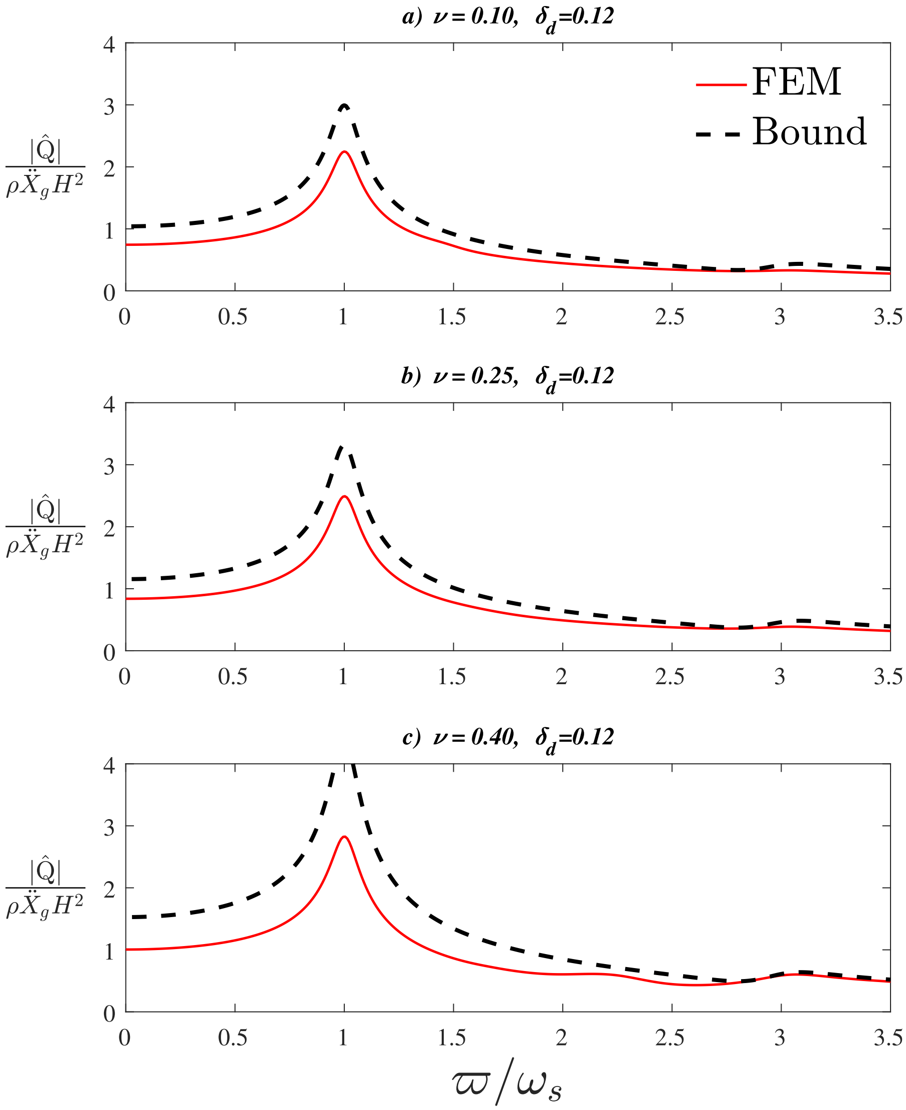

Fig. 4. Numerical results compared to bound Eq. (34). Poisson's ratio for sensitivity for constant damping $\delta_d = 0.12$.

range of Poisson rations $\nu = 0.2 - 0.4$, where, except for saturated clays (Verbrugge and Schroeder, 2018), most soils lie, the bound performs satisfactorily. In the limit of $\nu = 0$, the bound reduces to $\rho \ddot{X}_g H^2$. This thrust is interpreted in the literature as the inertial force exerted on a square soil mass of dimensions $H \times H$ and unit thickness (Kloukinas et al., 2012; Veletsos and Younan, 1994a).

5. Concluding remarks

We have presented a derivation of dynamic path-independent integrals in the frequency domain, followed by an application to the problem of assessing the earth thrust on a retaining wall in the configuration first introduced in Matsuo and Ohara (1960). The dynamic path-independent integrals have been utilized to obtain upper bounds of the earth thrust. Comparisons with finite element calculations attest to the good accuracy of the bounds, especially at high frequencies and compressibilities. The bounds improve on previous work by Veletsos and Younan (1994a,b), Arias et al. (1981) and Kloukinas et al. (2012), who resort to sundry *ad hoc* simplifications of the problem aimed at facilitating its analytical treatment. By contrast, the bounds presented in this work derive rigorously from the path-independence of the frequency-domain dynamic J-integral and are *ansatz*-free, follow from the full formulation of the problem without recourse to simplifying assumptions. The compact closed-form expressions that are obtained owe to the simplicity of the unidimensional fields in the far field.

The frequency-domain dynamic J-integral derived in this work, and extensions thereof in the spirit of the L and M integrals of static elasticity, can be applied to a range of other problems, including pipelines, landslides, tunnels, foundations and others.

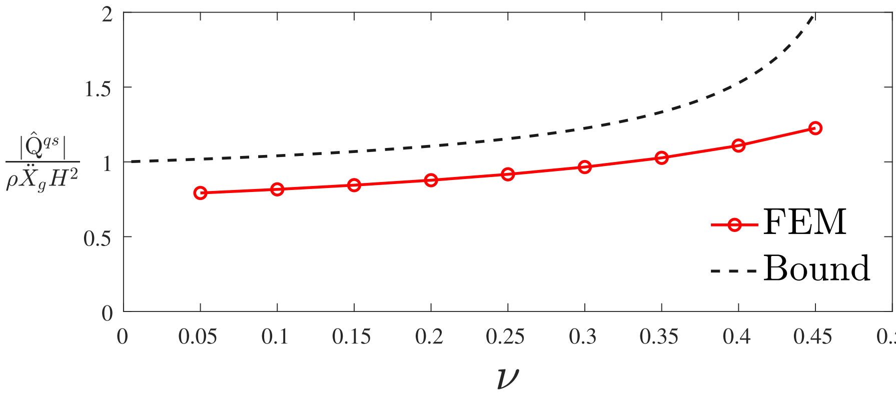

Fig. 5. Numerical results (red with circular markers) compared to bound Eq. (35) (black dashed), to visualize the effect of compressibility on the bound.

The path integrals can also be extended to layered or stratified soils, either continuously graded or with well-defined interfaces, an extension that is relevant to geomechanics, cf. Brandenberg et al. (2017). Consideration of constitutive behavior other than linear elasticity, such as nonlinear elasticity and poromechanics (Coussy, 2004), is also straightforward within the theory. As demonstrated in this paper, these and other extensions of the theory can be expected to supply useful insights and rigorous results in support of engineering practice.

CRediT authorship contribution statement

J. Garcia-Suarez: Conceptualization, Formal analysis, Writing - original draft, Validation. **D. Asimaki:** Supervision, Writing - review & editing. **M. Ortiz:** Supervision, Conceptualization, Formal analysis, Supervision, Writing - review & editing.

Acknowledgments

We would like to thank the anonymous reviewer whose comments helped to improve the paper. A portion of this work was developed during the doctoral studies of the first author, which were sponsored partially by a *Talentia Scholarship* of the regional government of Andalusia (Spain), and by the *Tyson Graduate Fellowship*, gifted by the Tyson family.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Supplementary material

A *Mathematica* notebook (Wolfram, 2000) containing the computations leading to results shown in the text can be found in the repository named `J_geotechnical` in the first author GitHub page github.com/jgarciasuarez.

References

Abeyaratne, Rohan, Knowles, James K., 2006. Evolution of Phase Transitions: A Continuum Theory. Cambridge University Press.
Arias, A., Sanchez-Sesma, F.J., Ovando-Shelley, E., 1981. A Simplified Elastic Model for Seismic Analysis of Earth-Retaining Structures with Limited Displacements. University of Missouri–Rolla.
Ben-Menahem, Ari, Singh, Sarva Jit, 2012. Seismic Waves and Sources. Springer Science & Business Media.
Brandenberg, Scott J., Mylonakis, George, Stewart, Jonathan P., 2017. Approximate solution for seismic earth pressures on rigid walls retaining inhomogeneous elastic soil. Soil Dyn. Earthq. Eng. 97, 468–477.
Byers, Nina, 1998. Noether's discovery of the deep connection between symmetries and conservation laws. ArXiv preprint physics/9807044.
Coussy, Olivier, 2004. Poromechanics. John Wiley & Sons.
Eshelby, J.D., 1956. The continuum theory of lattice defects. In: Solid State Physics, vol. 3. Elsevier, pp. 79–144.
Fletcher, Donnie C., 1976. Conservation laws in linear elastodynamics. Arch. Ration. Mech. Anal. 60 (4), 329–353.
Garcia-Suarez, Joaquin, Asimaki, Domniki, 2020. Exact seismic response of smooth rigid retaining walls resting on stiff soil. Int. J. Numer. Anal. Methods Geomech. 44 (13), 1750–1769.
Gibson, R.E., 1974. The analytical method in soil mechanics. Geotechnique 24 (2), 115–140.
Gurtin, Morton E., 1999. Configurational Forces as Basic Concepts of Continuum Physics, vol. 137. Springer Science & Business Media.
Kloukinas, Panos, Langousis, Miltiadis, Mylonakis, George, 2012. Simple wave solution for seismic earth pressures on non-yielding walls. J. Geotech. Geoenv. Eng. 138, 1514–1519.
Knowles, James K., Sternberg, Eli, 1972. On a class of conservation laws in linearized and finite elastostatics. Arch. Rational Mech. Anal. 44 (3), 187–211.

Kramer, Steven L., 1996. Geotechnical Earthquake Engineering. in Prentice–Hall International Series in Civil Engineering and Engineering Mechanics. Prentice Hall Inc., Englewood Cliffs, New Jersey.
Marsden, Jerrold E., Hughes, Thomas J.R., 1994. Mathematical Foundations of Elasticity. Courier Corporation.
Matsuo, H., Ohara, Sukeo, 1960. Lateral earth pressure and stability of quay walls during earthquakes. In: Proceedings of the Second World Conference on Earthquake Engineering, vol. 1. Science Council of Japan Tokyo-Kyoto, Japan, pp. 165–181.
Maugin, Gérard A., 2016. Configurational forces: Thermomechanics, Physics, Mathematics, and Numerics. Chapman and Hall/CRC.
NEHRP, 2001. NEHRP Recommended Provisions for Seismic Regulations for New Buildings and Other Structures. Building Seismic Safety Council.
Ostadan, Farhang, 2005. Seismic soil pressure for building walls: An updated approach. Soil Dyn. Earthq. Eng. 25 (7–10), 785–793.
Rice, James R., 1968. A path independent integral and the approximate analysis of strain concentration by notches and cracks. J. Appl. Mech. 35 (2), 379–386.
Rice, James R., 1985. Conserved integrals and energetic forces. In: Fundamentals of Deformation and Fracture (Eshelby Memorial Symposium). Cambridge University Press, Cambridge, pp. 33–56.
Richards, Jr., R., Elms, D.G., Budhu, M., 1990. Dynamic fluidization of soils. J. Geotech. Eng. 116 (5), 740–759.
Simulia, Dassault Systemes, 2010. Abaqus analysis user's manual. Dassault Systemes. Pawtucket, USA.
Veletsos, Anestis S., Younan, Adel H., 1994a. Dynamic soil pressures on rigid vertical walls. Earthq. Eng. Struct. Dyn. 23 (3), 275–301.
Veletsos, Anestis S., Younan, Adel H., 1994b. Dynamic modeling and response of soil-wall systems. J. Geotech. Eng. 120 (12), 2155–2179.
Verbrugge, Jean-Claude, Schroeder, Christian, 2018. Geotechnical Correlations for Soils and Rocks. Wiley Online Library.
Vlasov, V.Z., 1966. UN Leontiev, beams, plates, and shells on elastic foundation (translated from Russian). NASA TT F-357.
Wolfram, Stephen, 2000. The Mathematica Book, vol. 4. Cambridge University Press Cambridge.
Wood, John H., 1973. Earthquake-Induced Soil Pressures on Structures. California Institute of Technology.